

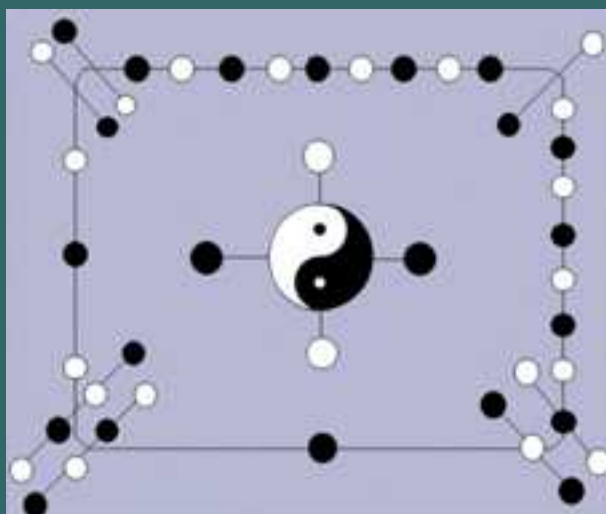
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Edited By Linfan MAO



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Only those who dare to fail greatly can ever achieve greatly.

By John Kennedy, the 35th President of the United States.

Neutrosophic Rings II

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Abstract: This paper is the continuation of the work started in [12]. The present paper is devoted to the study of ideals of neutrosophic rings. Neutrosophic quotient rings are also studied.

Key Words: Neutrosophic ring, neutrosophic ideal, pseudo neutrosophic ideal, neutrosophic quotient ring.

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§1. Introduction

The concept of neutrosophic rings was introduced by Vasantha Kandasamy and Florentin Smarandache in [1] where neutrosophic polynomial rings, neutrosophic matrix rings, neutrosophic direct product rings, neutrosophic integral domains, neutrosophic unique factorization domains, neutrosophic division rings, neutrosophic integral quaternions, neutrosophic rings of real quaternions, neutrosophic group rings and neutrosophic semigroup rings were studied. In [12], Agboola et al further studied neutrosophic rings. The structure of neutrosophic polynomial rings was presented. It was shown that *division algorithm* is generally not true for neutrosophic polynomial rings and it was also shown that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain. Also in [12], it was shown that $\langle R \cup I \rangle [x]$ cannot be a *unique factorization domain* even if R is a unique factorization domain and it was also shown that every non-zero neutrosophic principal ideal in a neutrosophic polynomial ring is not a neutrosophic prime ideal. The present paper is however devoted to the study of ideals of neutrosophic rings and neutrosophic quotient rings are also studied.

§2. Preliminaries and Results

For details about neutrosophy and neutrosophic rings, the reader should see [1] and [12].

Definition 2.1 Let $(R, +, \cdot)$ be any ring. The set

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I under the operations of R , where I is the neutrosophic element and $I^2 = I$.

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If $\langle R \cup I \rangle = \langle \mathbb{Z}_n \cup I \rangle$ with $n < \infty$, then $o(\langle \mathbb{Z}_n \cup I \rangle) = n^2$. Such a $\langle R \cup I \rangle$ is said to be a commutative neutrosophic ring with unity if $rs = sr$ for all $r, s \in \langle R \cup I \rangle$ and $1 \in \langle R \cup I \rangle$.

Definition 2.2 Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring of $\langle R \cup I \rangle$ if $P = \langle S \cup nI \rangle$, where S is a subring of R and n an integer, P is said to be generated by S and nI under the operations of R .

Definition 2.3 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$ which is just a ring. Then P is called a subring.

Definition 2.4 Let T be a non-empty set together with two binary operations $+$ and \cdot . T is said to be a pseudo neutrosophic ring if the following conditions hold:

- (1) T contains elements of the form $a + bI$, where a and b are real numbers and $b \neq 0$ for at least one value;
- (2) $(T, +)$ is an abelian group;
- (3) (T, \cdot) is a semigroup;
- (4) $\forall x, y, z \in T$, $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.

Definition 2.5 Let $\langle R \cup I \rangle$ be any neutrosophic ring. A non-empty subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions hold:

- (1) P is a neutrosophic subring of $\langle R \cup I \rangle$;
- (2) for every $p \in P$ and $r \in \langle R \cup I \rangle$, $rp \in P$ and $pr \in P$.

If only $rp \in P$, we call P a left neutrosophic ideal and if only $pr \in P$, we call P a right neutrosophic ideal. When $\langle R \cup I \rangle$ is commutative, there is no distinction between rp and pr and therefore P is called a left and right neutrosophic ideal or simply a neutrosophic ideal of $\langle R \cup I \rangle$.

Definition 2.6 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic subring of $\langle R \cup I \rangle$. P is said to be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$ if $\forall p \in P$ and $r \in \langle R \cup I \rangle$, $rp, pr \in P$.

Example 2.7 Let $\langle \mathbb{Z} \cup I \rangle$ be a neutrosophic ring of integers and let $P = \langle n\mathbb{Z} \cup I \rangle$ for a positive integer n . Then P is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$.

Example 2.8 Let $\langle R \cup I \rangle = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x, y, z \in \langle \mathcal{R} \cup I \rangle \right\}$ be the neutrosophic ring of 2×2 matrices and let $P = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \langle \mathcal{R} \cup I \rangle \right\}$. Then P is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$.

Theorem 2.9 Let $\langle \mathbb{Z}_p \cup I \rangle$ be a neutrosophic ring of integers modulo p , where p is a prime number. Then:

- (1) $\langle \mathbb{Z}_p \cup I \rangle$ has no neutrosophic ideals and
- (2) $\langle \mathbb{Z}_p \cup I \rangle$ has only one pseudo neutrosophic ideal of order p .

Proposition 2.10 *Let P , J and Q be neutrosophic ideals (resp. pseudo neutrosophic ideals) of a neutrosophic ring $\langle \mathcal{R} \cup I \rangle$. Then*

- (1) $P + J$ is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle \mathcal{R} \cup I \rangle$;
- (2) PJ is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle \mathcal{R} \cup I \rangle$;
- (3) $P \cap J$ is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle \mathcal{R} \cup I \rangle$;
- (4) $P(JQ) = (PJ)Q$;
- (5) $P(J + Q) = PJ + PQ$;
- (6) $(J + Q)P = JP + QP$.

Proof The proof is the same as in the classical ring. \square

Proposition 2.11 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring and let P be a subset of $\langle \mathcal{R} \cup I \rangle$. Then P is a neutrosophic ideal (resp. pseudo neutrosophic ideal) iff the following conditions hold:*

- (1) $P \neq \emptyset$;
- (2) $a, b \in P \Rightarrow a - b \in P$;
- (3) $a \in P, r \in \langle \mathcal{R} \cup I \rangle \Rightarrow ra, ar \in P$.

Proof the proof is the same as in the classical ring. \square

Proposition 2.12 *Let $\langle \mathcal{R} \cup I \rangle$ be any neutrosophic ring. Then $\langle \mathcal{R} \cup I \rangle$ and $\langle 0 \rangle$ are neutrosophic ideals of $\langle \mathcal{R} \cup I \rangle$.*

Proposition 2.13 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring with unity (no unit in $\langle \mathcal{R} \cup I \rangle$ since I^{-1} does not exist in $\langle \mathcal{R} \cup I \rangle$) and let P be a neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$. If $1 \in P$ then $P = \langle \mathcal{R} \cup I \rangle$.*

Proposition 2.14 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring with unity (no unit in $\langle \mathcal{R} \cup I \rangle$ since I^{-1} does not exist in $\langle \mathcal{R} \cup I \rangle$) and let P be a pseudo neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$. If $1 \in P$ then $P \neq \langle \mathcal{R} \cup I \rangle$.*

Proof Suppose that P is a pseudo neutrosophic ideal of the neutrosophic ring $\langle \mathcal{R} \cup I \rangle$ with unity and suppose that $1 \in P$. Let r be an arbitrary element of $\langle \mathcal{R} \cup I \rangle$. Then by the definition of P , $r.1 = r$ should be an element of P but since P is not a neutrosophic subring of $\langle \mathcal{R} \cup I \rangle$, there exist some elements $b = x + yI$ with $x, y \neq 0$ in $\langle \mathcal{R} \cup I \rangle$ which cannot be found in P . Hence $P \neq \langle \mathcal{R} \cup I \rangle$. \square

Proposition 2.15 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring and let $a = x + yI$ be a fixed element of $\langle \mathcal{R} \cup I \rangle$. Suppose that $P = \{ra : r \in \langle \mathcal{R} \cup I \rangle\}$ is a subset of $\langle \mathcal{R} \cup I \rangle$.*

- (1) *If $x, y \neq 0$, then P is a left neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$;*
- (2) *If $x = 0$, then P is a left pseudo neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$.*

Proof (1) is clear. For (2), if $x = 0$ then each element of P is of the form sI for some $s \in R$. Hence $P = \{0, sI\}$ which is a left pseudo neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$. \square

Theorem 2.16 *Every ideal of a neutrosophic ring $\langle \mathcal{R} \cup I \rangle$ is either neutrosophic or pseudo neutrosophic.*

Proof Suppose that P is any ideal of $\langle \mathcal{R} \cup I \rangle$. If $P \neq (0)$ or $P \neq \langle \mathcal{R} \cup I \rangle$, then there exists a subring S of R such that for a positive integer n , $P = \langle S \cup nI \rangle$. Let $p \in P$ and $r \in \langle \mathcal{R} \cup I \rangle$. By definition of P , $rp, pr \in P$ and the elements rp and pr are clearly of the form $a + bI$ with at least $b \neq 0$. \square

Definition 2.17 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring.*

- (1) *If P is a neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$ generated by an element $r = a + bI \in \langle \mathcal{R} \cup I \rangle$ with $a, b \neq 0$, then P is called a neutrosophic principal ideal of $\langle \mathcal{R} \cup I \rangle$, denoted by (r) .*
- (2) *If P is a pseudo neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$ generated by an element $r = aI \in \langle \mathcal{R} \cup I \rangle$ with $a \neq 0$, then P is called a pseudo neutrosophic principal ideal of $\langle \mathcal{R} \cup I \rangle$, denoted by (r) .*

Proposition 2.18 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring and let $r = a + bI \in \langle \mathcal{R} \cup I \rangle$ with $a, b \neq 0$.*

- (1) *(r) is the smallest neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$ containing r ;*
- (2) *Every pseudo neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$ is contained in some neutrosophic ideal of $\langle \mathcal{R} \cup I \rangle$.*

Proposition 2.19 *Every pseudo neutrosophic ideal of $\langle \mathcal{Z} \cup I \rangle$ is principal.*

Definition 2.20 *Let $\langle \mathcal{R} \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle \mathcal{R} \cup I \rangle$.*

- (1) *P is said to be maximal if for any neutrosophic ideal (resp. pseudo neutrosophic ideal) J of $\langle \mathcal{R} \cup I \rangle$ such that $P \subseteq J$ we have either $J = P$ or $J = \langle \mathcal{R} \cup I \rangle$.*
- (2) *P is said to be a prime ideal if for any two neutrosophic ideals (resp. pseudo neutrosophic ideals) J and Q of $\langle \mathcal{R} \cup I \rangle$ such that $JQ \subseteq P$ we have either $J \subseteq P$ or $Q \subseteq P$.*

Proposition 2.21 *Let $\langle \mathcal{R} \cup I \rangle$ be a commutative neutrosophic ring with unity and let P be a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle \mathcal{R} \cup I \rangle$. Then P is prime iff $xy \in P$ with x and y in $\langle \mathcal{R} \cup I \rangle$ implies that either $x \in P$ or $y \in P$.*

Example 2.22 In $\langle \mathcal{Z} \cup I \rangle$ the neutrosophic ring of integers:

- (1) (nI) where n is a positive integer is a pseudo neutrosophic principal ideal.
- (2) (I) is the only maximal pseudo neutrosophic ideal.
- (3) (0) is the only prime neutrosophic ideal (resp. prime pseudo neutrosophic ideal).

Definition 2.23 *Let $\langle \mathcal{R} \cup I \rangle$ be a commutative neutrosophic ring and let $x = a + bI$ be an element of $\langle \mathcal{R} \cup I \rangle$ with $a, b \in R$.*

- (1) *If $a, b \neq 0$ and there exists a positive integer n such that $x^n = 0$ then x is called a strong neutrosophic nilpotent element of $\langle \mathcal{R} \cup I \rangle$.*
- (2) *If $a = 0, b \neq 0$ and there exists a positive integer n such that $x^n = 0$ then x is called a weak neutrosophic nilpotent element of $\langle \mathcal{R} \cup I \rangle$.*

(3) If $b = 0$ and there exists a positive integer n such that $x^n = 0$ then x is called an ordinary nilpotent element of $\langle R \cup I \rangle$.

Example 2.24 In the neutrosophic ring $\langle \mathbb{Z}_4 \cup I \rangle$ of integers modulo 4, 0 and 2 are ordinary nilpotent elements, $2I$ is a weak neutrosophic nilpotent element and $2 + 2I$ is a strong neutrosophic nilpotent element.

Proposition 2.25 Let $\langle R \cup I \rangle$ be a commutative neutrosophic ring.

- (1) The set of all strong neutrosophic nilpotent elements of $\langle R \cup I \rangle$ is not a neutrosophic ideal.
- (2) The set of all weak neutrosophic nilpotent elements of $\langle R \cup I \rangle$ is not a neutrosophic ideal.
- (3) The set of all nilpotent (ordinary, strong and weak neutrosophic) elements of the commutative neutrosophic ring $\langle R \cup I \rangle$ is a neutrosophic ideal of $\langle R \cup I \rangle$.

Definition 2.26 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R \cup I \rangle$. Let $\langle R \cup I \rangle / P$ be a set defined by

$$\langle R \cup I \rangle / P = \{r + P : r \in \langle R \cup I \rangle\}.$$

If addition and multiplication in $\langle R \cup I \rangle / P$ are defined by

$$\begin{aligned} (r + P) + (s + P) &= (r + s) + P, \\ (r + P)(s + P) &= (rs) + P, r, s \in \langle R \cup I \rangle, \end{aligned}$$

it can be shown that $\langle R \cup I \rangle / P$ is a neutrosophic ring called the neutrosophic quotient ring with P as an additive identity.

Definition 2.27 Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a subset of $\langle R \cup I \rangle$.

- (1) If P is a neutrosophic ideal of $\langle R \cup I \rangle$ and $\langle R \cup I \rangle / P$ is just a ring, then $\langle R \cup I \rangle / P$ is called a false neutrosophic quotient ring;
- (2) If P is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$ and $\langle R \cup I \rangle / P$ is a neutrosophic ring, then $\langle R \cup I \rangle / P$ is called a pseudo neutrosophic quotient ring;
- (3) If P is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$ and $\langle R \cup I \rangle / P$ is just a ring, then $\langle R \cup I \rangle / P$ is called a false pseudo neutrosophic quotient ring.

Example 2.28 Let $\langle \mathbb{Z}_6 \cup I \rangle = \{0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 5I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 2 + 5I, 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 3 + 5I, 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, 4 + 5I, 5 + I, 5 + 2I, 5 + 3I, 5 + 4I, 5 + 5I\}$ be a neutrosophic ring of integers modulo 6.

(1) If $P = \{0, 2, I, 2I, 3I, 4I, 5I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 2 + 5I\}$, then P is a neutrosophic ideal of $\langle \mathbb{Z}_6 \cup I \rangle$ but $\langle \mathbb{Z}_6 \cup I \rangle / P = \{P, 1 + P, 3 + P, 4 + P, 5 + P\}$ is just a ring and thus $\langle \mathbb{Z}_6 \cup I \rangle / P$ is a false neutrosophic quotient ring.

(2) If $P = \{0, 2I, 4I\}$, then P is a pseudo neutrosophic ideal of $\langle \mathbb{Z}_6 \cup I \rangle$ and the quotient ring

$\langle \mathbb{Z}_6 \cup I \rangle / P = \{P, 1+P, 2+P, 3+P, 4+P, 5+P, I+P, (1+I)+P, (2+I)+P, (3+I)+P, (4+I)+P, (5+I)+P\}$ is a pseudo neutrosophic quotient ring.

(3) If $P = \{0, I, 2I, 3I, 4I, 5I\}$, then P is a pseudo neutrosophic ideal and the quotient ring. $\langle \mathbb{Z}_6 \cup I \rangle / P = \{P, 1+P, 2+P, 3+P, 4+P, 5+P\}$ is a false pseudo neutrosophic quotient ring.

Definition 2.29 Let $\langle R \cup I \rangle$ and $\langle S \cup I \rangle$ be any two neutrosophic rings. The mapping $\phi : \langle R \cup I \rangle \rightarrow \langle S \cup I \rangle$ is called a neutrosophic ring homomorphism if the following conditions hold:

- (1) ϕ is a ring homomorphism;
- (2) $\phi(I) = I$.

If in addition ϕ is both 1-1 and onto, then ϕ is called a neutrosophic isomorphism and we write $\langle R \cup I \rangle \cong \langle S \cup I \rangle$.

The set $\{x \in \langle R \cup I \rangle : \phi(x) = 0\}$ is called the kernel of ϕ and is denoted by $\text{Ker}\phi$.

Theorem 2.30 Let $\phi : \langle R \cup I \rangle \rightarrow \langle S \cup I \rangle$ be a neutrosophic ring homomorphism and let $K = \text{Ker}\phi$ be the kernel of ϕ . Then:

- (1) K is always a subring of $\langle R \cup I \rangle$;
- (2) K cannot be a neutrosophic subring of $\langle R \cup I \rangle$;
- (3) K cannot be an ideal of $\langle R \cup I \rangle$.

Example 2.31 Let $\langle \mathbb{Z} \cup I \rangle$ be a neutrosophic ring of integers and let $P = 5\mathbb{Z} \cup I$. It is clear that P is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$ and the neutrosophic quotient ring $\langle \mathbb{Z} \cup I \rangle / P$ is obtained as

$$\begin{aligned} \langle \mathbb{Z} \cup I \rangle / P = & \{P, 1+P, 2+P, 3+P, 4+P, I+P, 2I+P, 3I+P, 4I+P, \\ & (1+I)+P, (1+2I)+P, (1+3I)+P, (1+4I)+P, (2+I)+P, \\ & (2+2I)+P, (2+3I)+P, (2+4I)+P, (3+I)+P, (3+2I)+P, \\ & (3+3I)+P, (3+4I)+P, (4+I)+P, (4+2I)+P, (4+3I)+P, (4+4I)+P\}. \end{aligned}$$

The following can easily be deduced from the example:

- (1) $\langle \mathbb{Z} \cup I \rangle / P$ is neither a field nor an integral domain.
- (2) $\langle \mathbb{Z} \cup I \rangle / P$ and the neutrosophic ring $\langle \mathbb{Z}_5 \cup I \rangle$ of integers modulo 5 are structurally the same but then
- (3) The mapping $\phi : \langle \mathbb{Z} \cup I \rangle \rightarrow \langle \mathbb{Z} \cup I \rangle / P$ defined by $\phi(x) = x + P$ for all $x \in \langle \mathbb{Z} \cup I \rangle$ is not a neutrosophic ring homomorphism and consequently $\langle \mathbb{Z} \cup I \rangle \not\cong \langle \mathbb{Z} \cup I \rangle / P \not\cong \langle \mathbb{Z}_5 \cup I \rangle$.

These deductions are recorded in the next proposition.

Proposition 2.32 Let $\langle \mathbb{Z} \cup I \rangle$ be a neutrosophic ring of integers and let $P = \langle n\mathbb{Z} \cup I \rangle$ where n is a positive integer. Then:

- (1) $\langle \mathbb{Z} \cup I \rangle / P$ is a neutrosophic ring;
- (2) $\langle \mathbb{Z} \cup I \rangle / P$ is neither a field nor an integral domain even if n is a prime number;
- (3) $\langle \mathbb{Z} \cup I \rangle / P \not\cong \langle \mathbb{Z}_n \cup I \rangle$.

Theorem 2.33 *If P is a pseudo neutrosophic ideal of the neutrosophic ring $\langle \mathcal{Z}_n \cup I \rangle$ of integers modulo n , then*

$$\langle \mathcal{Z}_n \cup I \rangle / P \cong \mathcal{Z}_n.$$

Proof Let $P = \{0, I, 2I, 3I, \dots, (n-3)I, (n-2)I, (n-1)I\}$. It is clear that P is a pseudo neutrosophic ideal of $\langle \mathcal{Z}_n \cup I \rangle$ and $\langle \mathcal{Z}_n \cup I \rangle / P$ is a false neutrosophic quotient ring given by

$$\langle \mathcal{Z}_n \cup I \rangle / P = \{P, 1 + P, 2 + P, 3 + P, \dots, (n-3) + P, (n-2) + P, (n-1) + P\} \cong \mathcal{Z}_n. \quad \square$$

Proposition 2.34 *Let $\phi : \langle R \cup I \rangle \rightarrow \langle S \cup I \rangle$ be a neutrosophic ring homomorphism.*

- (1) *The set $\phi(\langle R \cup I \rangle) = \{\phi(r) : r \in \langle R \cup I \rangle\}$ is a neutrosophic subring of $\langle S \cup I \rangle$;*
- (2) *$\phi(-r) = -\phi(r) \quad \forall r \in \langle R \cup I \rangle$;*
- (3) *If 0 is the zero of $\langle R \cup I \rangle$, then $\phi(0)$ is the zero of $\phi(\langle R \cup I \rangle)$;*
- (4) *If P is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle R \cup I \rangle$, then $\phi(P)$ is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle S \cup I \rangle$;*
- (5) *If J is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle S \cup I \rangle$, then $\phi^{-1}(J)$ is a neutrosophic ideal (resp. pseudo neutrosophic ideal) of $\langle R \cup I \rangle$;*
- (6) *If $\langle R \cup I \rangle$ has unity 1 and $\phi(1) \neq 0$ in $\langle S \cup I \rangle$, then $\phi(1)$ is the unity of $\phi(\langle R \cup I \rangle)$;*
- (7) *If $\langle R \cup I \rangle$ is commutative, then $\phi(\langle R \cup I \rangle)$ is commutative.*

Proof The proof is the same as in the classical ring. \square

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Non-Solvable Spaces of Linear Equation Systems

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Abstract: A *Smarandache system* $(\Sigma; \mathcal{R})$ is such a mathematical system that has at least one *Smarandachely denied rule* in \mathcal{R} , i.e., there is a rule in $(\Sigma; \mathcal{R})$ that behaves in at least two different ways within the same set Σ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways. For such systems, the linear equation systems without solutions, i.e., non-solvable linear equation systems are the most simple one. We characterize such non-solvable linear equation systems with their homeomorphisms, particularly, the non-solvable linear equation systems with 2 or 3 variables by combinatorics. It is very interesting that every planar graph with each edge a straight segment is homologous to such a non-solvable linear equation with 2 variables.

Key Words: Smarandachely denied axiom, Smarandache system, non-solvable linear equations, \vee -solution, \wedge -solution.

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§1. Introduction

Finding the exact solution of equation system is a main but a difficult objective unless the case of linear equations in classical mathematics. Contrary to this fact, *what is about the non-solvable case?* In fact, such an equation system is nothing but a contradictory system, and characterized only by non-solvable equations for conclusion. But our world is overlap and hybrid. The number of non-solvable equations is more than that of the solvable. The main purpose of this paper is to characterize the behavior of such linear equation systems.

Let $\mathbb{R}^m, \mathbb{R}^n$ be Euclidean spaces with dimensional $m, n \geq 1$ and $T: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a $\mathbb{C}^k, 1 \leq k \leq \infty$ function such that $T(\bar{x}_0, \bar{y}_0) = \bar{0}$ for $\bar{x}_0 \in \mathbb{R}^n, \bar{y}_0 \in \mathbb{R}^m$ and the $m \times m$ matrix $\partial T^j / \partial y^i(\bar{x}_0, \bar{y}_0)$ is non-singular, i.e.,

$$\det\left(\frac{\partial T^j}{\partial y^i}\right)\Big|_{(\bar{x}_0, \bar{y}_0)} \neq 0, \text{ where } 1 \leq i, j \leq m.$$

Then the implicit function theorem ([1]) implies that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\bar{x}_0, W \subset \mathbb{R}^m$ of \bar{y}_0 and a \mathbb{C}^k function $\phi: V \rightarrow W$ such that

$$T(\bar{x}, \phi(\bar{x})) = \bar{0}.$$

Thus there always exists solutions for the equation $T(\bar{x}, \overline{\phi(\bar{x})}) = \bar{0}$ if T is $\mathbb{C}^k, 1 \leq k \leq \infty$. Now let $T_1, T_2, \dots, T_m, m \geq 1$ be different \mathbb{C}^k functions $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ for an integer $k \geq 1$. An

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equation system discussed in this paper is with the form following

$$T_i(\bar{x}, \bar{y}) = \bar{0}, \quad 1 \leq i \leq m. \quad (Eq)$$

A point (\bar{x}_0, \bar{y}_0) is a \vee -solution of the equation system (Eq) if

$$T_i(\bar{x}_0, \bar{y}_0) = \bar{0}$$

for some integers i , $1 \leq i \leq m$, and a \wedge -solution of (Eq) if

$$T_i(\bar{x}_0, \bar{y}_0) = \bar{0}$$

for all integers $1 \leq i \leq m$. Denoted by S_i^0 the solutions of equation $T_i(\bar{x}, \bar{y}) = \bar{0}$ for integers $1 \leq i \leq m$. Then $\bigcup_{i=1}^m S_i^0$ and $\bigcap_{i=1}^m S_i^0$ are respectively the \vee -solutions and \wedge -solutions of equations (Eq). By definition, we are easily knowing that the \wedge -solution is nothing but the same as the classical solution.

Definition 1.1 The \vee -solvable, \wedge -solvable and non-solvable spaces of equations (Eq) are respectively defined by

$$\bigcup_{i=1}^m S_i^0, \quad \bigcap_{i=1}^m S_i^0 \quad \text{and} \quad \bigcup_{i=1}^m S_i^0 - \bigcap_{i=1}^m S_i^0.$$

Now we construct a finite graph $G[Eq]$ of equations (Eq) following:

$$V(G[Eq]) = \{v_i | 1 \leq i \leq m\},$$

$$E(G[Eq]) = \{(v_i, v_j) | \exists (\bar{x}_0, \bar{y}_0) \Rightarrow T_i(\bar{x}_0, \bar{y}_0) = \bar{0} \wedge T_j(\bar{x}_0, \bar{y}_0) = \bar{0}, \quad 1 \leq i, j \leq m\}.$$

Such a graph $G[Eq]$ can be also represented by a vertex-edge labeled graph $G^L[Eq]$ following:

$$V(G^L[Eq]) = \{S_i^0 | 1 \leq i \leq m\},$$

$$E(G[Eq]) = \{(S_i^0, S_j^0) \text{ labeled with } S_i^0 \cap S_j^0 | S_i^0 \cap S_j^0 \neq \emptyset, 1 \leq i, j \leq m\}.$$

For example, let $S_1^0 = \{a, b, c\}$, $S_2^0 = \{c, d, e\}$, $S_3^0 = \{a, c, e\}$ and $S_4^0 = \{d, e, f\}$. Then its edge-labeled graph $G[Eq]$ is shown in Fig.1 following.

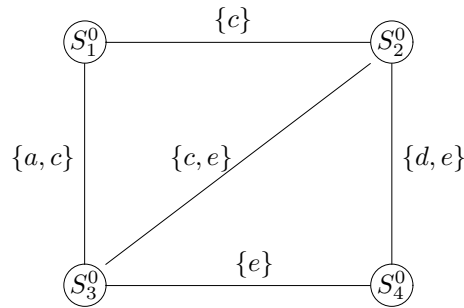


Fig.1

Notice that $\bigcup_{i=1}^m S_i^0 = \bigcup_{i=1}^m S_i^0$, i.e., the non-solvable space is empty only if $m = 1$ in (Eq). Generally, let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where \mathcal{R}_i is a rule on Σ_i for integers $1 \leq i \leq m$. If for two integers i, j , $1 \leq i, j \leq m$, $\Sigma_i \neq \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \neq \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*.

Definition 1.2([12]-[13]) *A rule in \mathcal{R} a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in \mathcal{R} .

Thus, such a Smarandache system is a contradictory system. Generally, we know the conception of Smarandache multi-space with its underlying combinatorial structure defined following.

Definition 1.3([8]-[10]) *Let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\tilde{\Sigma}$ is a union $\bigcup_{i=1}^m \Sigma_i$ with rules $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$ on $\tilde{\Sigma}$, i.e., the rule \mathcal{R}_i on Σ_i for integers $1 \leq i \leq m$, denoted by $(\tilde{\Sigma}; \tilde{\mathcal{R}})$.*

Similarly, the underlying graph of a Smarandache multi-space $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ is an edge-labeled graph defined following.

Definition 1.4([8]-[10]) *Let $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ be a Smarandache multi-space with $\tilde{\Sigma} = \bigcup_{i=1}^m \Sigma_i$ and $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$. Its underlying graph $G[\tilde{\Sigma}, \tilde{\mathcal{R}}]$ is defined by*

$$\begin{aligned} V(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) &= \{\Sigma_1, \Sigma_2, \dots, \Sigma_m\}, \\ E(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) &= \{(\Sigma_i, \Sigma_j) \mid \Sigma_i \cap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\} \end{aligned}$$

with an edge labeling

$$l^E : (\Sigma_i, \Sigma_j) \in E(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) \rightarrow l^E(\Sigma_i, \Sigma_j) = \varpi(\Sigma_i \cap \Sigma_j),$$

where ϖ is a characteristic on $\Sigma_i \cap \Sigma_j$ such that $\Sigma_i \cap \Sigma_j$ is isomorphic to $\Sigma_k \cap \Sigma_l$ if and only if $\varpi(\Sigma_i \cap \Sigma_j) = \varpi(\Sigma_k \cap \Sigma_l)$ for integers $1 \leq i, j, k, l \leq m$.

We consider the simplest case, i.e., all equations in (Eq) are linear with integers $m \geq n$ and $m, n \geq 1$ in this paper because we are easily know the necessary and sufficient condition of a linear equation system is solvable or not in linear algebra. For terminologies and notations not mentioned here, we follow [2]-[3] for linear algebra, [8] and [10] for graphs and topology.

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq)$$

be a linear equation system with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers $m, n \geq 1$. Define an augmented matrix A^+ of A by $(b_1, b_2, \dots, b_m)^T$ following:

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

We assume that all equations in (LEq) are non-trivial, i.e., there are no numbers λ such that

$$(a_{i1}, a_{i2}, \dots, a_{in}, b_i) = \lambda(a_{j1}, a_{j2}, \dots, a_{jn}, b_j)$$

for any integers $1 \leq i, j \leq m$. Such a linear equation system (LEq) is *non-solvable* if there are no solutions $x_i, 1 \leq i \leq n$ satisfying (LEq) .

§2. A Necessary and Sufficient Condition for Non-Solvable Linear Equations

The following result on non-solvable linear equations is well-known in linear algebra([2]-[3]).

Theorem 2.1 *The linear equation system (LEq) is solvable if and only if $\text{rank}(A) = \text{rank}(A^+)$. Thus, the equation system (LEq) is non-solvable if and only if $\text{rank}(A) \neq \text{rank}(A^+)$.*

We introduce the conception of parallel linear equations following.

Definition 2.2 *For any integers $1 \leq i, j \leq m, i \neq j$, the linear equations*

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

are called parallel if there exists a constant c such that

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

Then we know the following conclusion by Theorem 2.1.

Corollary 2.3 *For any integers $i, j, i \neq j$, the linear equation system*

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \end{cases}$$

is non-solvable if and only if they are parallel.

Proof By Theorem 2.1, we know that the linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n &= b_j \end{aligned}$$

is non-solvable if and only if $\text{rank}A' \neq \text{rank}B'$, where

$$A' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix}, \quad B' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} & b_1 \\ a_{j1} & a_{j2} & \cdots & a_{jn} & b_2 \end{bmatrix}.$$

It is clear that $1 \leq \text{rank}A' \leq \text{rank}B' \leq 2$ by the definition of matrixes A' and B' . Consequently, $\text{rank}A' = 1$ and $\text{rank}B' = 2$. Thus the matrix A' , B' are respectively elementary equivalent to matrixes

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}.$$

i.e., there exists a constant c such that $c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in}$ but $c \neq b_j/b_i$. Whence, the linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n &= b_j \end{aligned}$$

is parallel by definition. □

We are easily getting another necessary and sufficient condition for non-solvable linear equations (LEq) by three elementary transformations on a $m \times (n+1)$ matrix A^+ defined following:

- (1) *Multiplying one row of A^+ by a non-zero scalar c ;*
- (2) *Replacing the i th row of A^+ by row i plus a non-zero scalar c times row j ;*
- (3) *Interchange of two row of A^+ .*

Such a transformation naturally induces a transformation of linear equation system (LEq), denoted by $T(LEq)$. By applying Theorem 2.1, we get a generalization of Corollary 2.3 for non-solvable linear equation system (LEq) following.

Theorem 2.4 *A linear equation system (LEq) is non-solvable if and only if there exists a composition T of series elementary transformations on A^+ with $T(A^+)$ the forms following*

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}$$

Proof Let $T(A^+)$ be

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}.$$

If there are integers $1 \leq i, j \leq m$ such that the linear equations

$$\begin{aligned} a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n &= b'_i, \\ a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n &= b'_j \end{aligned}$$

are parallel, then there must be $S_i^0 \cap S_j^0 = \emptyset$, where S_i^0, S_j^0 are respectively the solutions of linear equations $a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i$ and $a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$. Whence, there are no edges (S_i^0, S_j^0) in $G[LEq]$ by definition. Thus $G[LEq] \not\simeq K_m$. \square

We wish to find conditions for non-solvable linear equation systems (LEq) without elementary transformations. In fact, we are easily determining $G[LEq]$ of a linear equation system (LEq) by Corollary 2.3. Let L_i be the i th linear equation. By Corollary 2.3, we divide these equations $L_i, 1 \leq i \leq m$ into *parallel families*

$$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$$

by the property that all equations in a family \mathcal{C}_i are parallel and there are no other equations parallel to lines in \mathcal{C}_i for integers $1 \leq i \leq s$. Denoted by $|\mathcal{C}_i| = n_i, 1 \leq i \leq s$. Then the following conclusion is clear by definition.

Theorem 2.6 *Let (LEq) be a linear equation system for integers $m, n \geq 1$. Then*

$$G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$$

with $n_1 + n_2 + \cdots + n_s = m$, where \mathcal{C}_i is the parallel family with $n_i = |\mathcal{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) and (LEq) is non-solvable if $s \geq 2$.

Proof Notice that equations in a family \mathcal{C}_i is parallel for an integer $1 \leq i \leq m$ and each of them is not parallel with all equations in $\bigcup_{1 \leq l \leq m, l \neq i} \mathcal{C}_l$. Let $n_i = |\mathcal{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) . By definition, we know

$$G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$$

with $n_1 + n_2 + \cdots + n_s = m$.

Notice that the linear equation system (LEq) is solvable only if $G[LEq] \simeq K_m$ by definition. Thus the linear equation system (LEq) is non-solvable if $s \geq 2$. \square

Notice that the conditions in Theorem 2.6 is not sufficient, i.e., if $G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$, we can not claim that (LEq) is non-solvable or not. For example, let (LEq^*) and (LEq^{**}) be

two linear equations systems with

$$A_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad A_2^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$

Then $G[LEq^*] \simeq G[LEq^{**}] \simeq K_4$. Clearly, the linear equation system (LEq^*) is solvable with $x_1 = 0, x_2 = 0$ but (LEq^{**}) is non-solvable. We will find necessary and sufficient conditions for linear equation systems with two or three variables just by their combinatorial structures in the following sections.

§3. Linear Equation System with 2 Variables

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq2)$$

be a linear equation system in 2 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \dots & \dots \\ a_{m1} & a_{m2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for an integer $m \geq 2$. Then Theorem 2.4 is refined in the following.

Theorem 3.1 *A linear equation system (LEq2) is non-solvable if and only if one of the following conditions hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$;*
- (2) *there are integers $1 \leq i, j, k \leq m$ such that*

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}}.$$

Proof The condition (1) is nothing but the conclusion in Corollary 2.3, i.e., the i th equation is parallel to the j th equation. Now if there no such parallel equations in $(LEq2)$, let T be the elementary transformation replacing all other j th equations by the j th equation plus $(-a_{j1}/a_{i1})$

times the i th equation for integers $1 \leq j \leq m$. We get a transformation $T(A^+)$ of A^+ following

$$T(A^+) = \left[\begin{array}{c|cc|cc} 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{11} & a_{12} & a_{11} & b_1 \\ \dots & & \dots & & \dots \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{s1} & a_{s2} & a_{s1} & b_s \\ a_{i1} & & a_{i2} & & b_i \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{t1} & a_{t2} & a_{t1} & b_t \\ \dots & & \dots & & \dots \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{m1} & a_{m2} & a_{m1} & b_m \end{array} \right],$$

where $s = i - 1$, $t = i + 1$. Applying Corollary 2.3 again, we know that there are integers $1 \leq i, j, k \leq m$ such that

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}}.$$

if the linear equation system (LEQ2) is non-solvable. \square

Notice that a linear equation $ax_1 + bx_2 = c$ with $a \neq 0$ or $b \neq 0$ is a straight line on \mathbb{R}^2 . We get the following result.

Theorem 3.2 *A linear equation system (LEq2) is non-solvable if and only if one of conditions following hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$;*
- (2) *let $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ and*

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Then there is an integer i , $1 \leq i \leq m$ such that

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \neq 0.$$

Proof If the linear equation system ($LEq2$) has a solution (x_1^0, x_2^0) , then

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and $a_{i1}x_1^0 + a_{i2}x_2^0 = b_i$, i.e., $a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) = 0$ for any integers $1 \leq i \leq m$. Thus, if the linear equation system ($LEq2$) is non-solvable, there must be integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$, or there is an integer $1 \leq i \leq m$ such that

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \neq 0.$$

This completes the proof. \square

For a non-solvable linear equation system ($LEq2$), there is a naturally induced *intersection-free graph* $I[LEq2]$ by ($LEq2$) on the plane \mathbb{R}^2 defined following:

$$V(I[LEq2]) = \{(x_1^{ij}, x_2^{ij}) | a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i, a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j, 1 \leq i, j \leq m\}.$$

$E(I[LEq2]) = \{(v_{ij}, v_{il}) | \text{the segment between points } (x_1^{ij}, x_2^{ij}) \text{ and } (x_1^{il}, x_2^{il}) \text{ in } \mathbb{R}^2\}$. (where $v_{ij} = (x_1^{ij}, x_2^{ij})$ for $1 \leq i, j \leq m$).

Such an intersection-free graph is clearly a planar graph with each edge a straight segment since all intersection of edges appear at vertices. For example, let the linear equation system be ($LEq2$) with

$$A^+ = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}.$$

Then its intersection-free graph $I[LEq2]$ is shown in Fig.2.

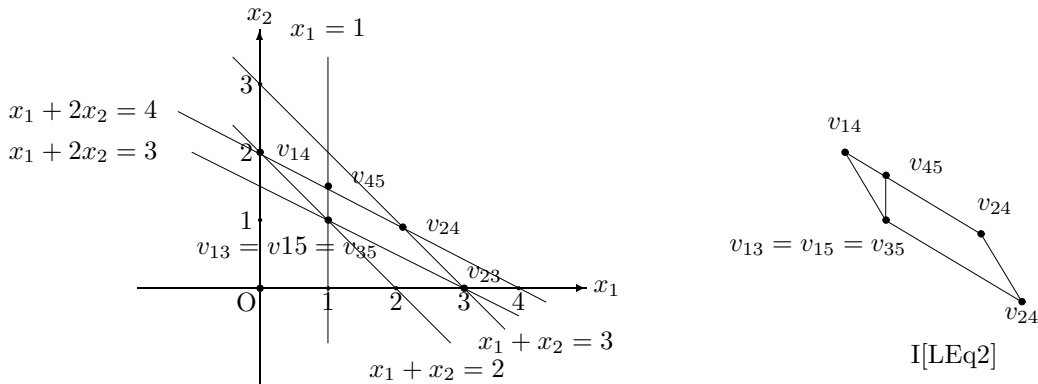


Fig.2

Let H be a planar graph with each edge a straight segment on \mathbb{R}^2 . Its c -line graph $L_C(H)$ is defined by

$$\begin{aligned} V(L_C(H)) &= \{\text{straight lines } L = e_1 e_2 \cdots e_l, l \geq 1 \text{ in } H\}; \\ E(L_C(H)) &= \{(L_1, L_2) \mid \text{if } e_i^1 \text{ and } e_j^2 \text{ are adjacent in } H \text{ for } L_1 = e_1^1 e_2^1 \cdots e_l^1, L_2 = e_1^2 e_2^2 \cdots e_s^2, l, s \geq 1\}. \end{aligned}$$

The following result characterizes the combinatorial structure of non-solvable linear equation systems with two variables by intersection-free graphs $I[LEq2]$.

Theorem 3.3 *A linear equation system (LEq2) is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H),$$

where H is a planar graph of order $|H| \geq 2$ on \mathbb{R}^2 with each edge a straight segment

Proof Notice that there is naturally a one to one mapping $\phi : V(G[LEq2]) \rightarrow V(L_C(I[LEq2]))$ determined by $\phi(S_i^0) = S_i^1$ for integers $1 \leq i \leq m$, where S_i^0 and S_i^1 denote respectively the solutions of equation $a_{i1}x_1 + a_{i2}x_2 = b_i$ on the plane \mathbb{R}^2 or the union of points between (x_1^{ij}, x_2^{ij}) and (x_1^{il}, x_2^{il}) with

$$\begin{cases} a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i \\ a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j \end{cases}$$

and

$$\begin{cases} a_{i1}x_1^{il} + a_{i2}x_2^{il} = b_i \\ a_{l1}x_1^{il} + a_{l2}x_2^{il} = b_l \end{cases}$$

for integers $1 \leq i, j, l \leq m$. Now if $(S_i^0, S_j^0) \in E(G[LEq2])$, then $S_i^0 \cap S_j^0 \neq \emptyset$. Whence,

$$S_i^1 \cap S_j^1 = \phi(S_i^0) \cap \phi(S_j^0) = \phi(S_i^0 \cap S_j^0) \neq \emptyset$$

by definition. Thus $(S_i^1, S_j^1) \in L_C(I[LEq2])$. By definition, ϕ is an isomorphism between $G[LEq2]$ and $L_C(I[LEq2])$, a line graph of planar graph $I[LEq2]$ with each edge a straight segment.

Conversely, let H be a planar graph with each edge a straight segment on the plane \mathbb{R}^2 . Not loss of generality, we assume that edges $e_{1,2}, \dots, e_l \in E(H)$ is on a straight line L with equation $a_{L1}x_1 + a_{L2}x_2 = b_L$. Denote all straight lines in H by \mathcal{C} . Then H is the intersection-free graph of linear equation system

$$a_{L1}x_1 + a_{L2}x_2 = b_L, \quad L \in \mathcal{C}. \quad (LEq2^*)$$

Thus,

$$G[LEq2^*] \simeq H.$$

This completes the proof. \square

Similarly, we can also consider the liner equation system $(LEq2)$ with condition on x_1 or x_2 such as

$$AX = (b_1, b_2, \dots, b_m)^T \quad (L^-Eq2)$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \dots & \dots \\ a_{m1} & a_{m2} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and $x_1 \geq x^0$ for a real number x^0 and an integer $m \geq 2$. In geometry, each of there equation is a ray on the plane \mathbb{R}^2 , seeing also references [5]-[6]. Then the following conclusion can be obtained like with Theorems 3.2 and 3.3.

Theorem 3.4 *A linear equation system (L-Eq2) is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H),$$

where H is a planar graph of order $|H| \geq 2$ on \mathbb{R}^2 with each edge a straight segment.

§4. Linear Equation Systems with 3 Variables

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq3)$$

be a linear equation system in 3 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for an integer $m \geq 3$. Then Theorem 2.4 is refined in the following.

Theorem 4.1 *A linear equation system (LEq3) is non-solvable if and only if one of the following conditions hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3} \neq b_i/b_j$;*
- (2) *if (a_{i1}, a_{i2}, a_{i3}) and (a_{j1}, a_{j2}, a_{j3}) are independent, then there are numbers λ, μ and an integer l , $1 \leq l \leq m$ such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$;

- (3) *if (a_{i1}, a_{i2}, a_{i3}) , (a_{j1}, a_{j2}, a_{j3}) and (a_{k1}, a_{k2}, a_{k3}) are independent, then there are numbers λ, μ, ν and an integer l , $1 \leq l \leq m$ such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$.

Proof By Theorem 2.1, the linear equation system (*LEq3*) is non-solvable if and only if $1 \leq \text{rank}A \neq \text{rank}A^+ \leq 4$. Thus the non-solvable possibilities of (*LEq3*) are respectively $\text{rank}A = 1, 2 \leq \text{rank}A^+ \leq 4, \text{rank}A = 2, 3 \leq \text{rank}A^+ \leq 4$ and $\text{rank}A = 3, \text{rank}A^+ = 4$. We discuss each of these cases following.

Case 1 $\text{rank}A = 1$ but $2 \leq \text{rank}A^+ \leq 4$

In this case, all row vectors in A are dependent. Thus there exists a number λ such that $\lambda = a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3}$ but $\lambda \neq b_i/b_j$.

Case 2 $\text{rank}A = 2, 3 \leq \text{rank}A^+ \leq 4$

In this case, there are two independent row vectors. Without loss of generality, let (a_{i1}, a_{i2}, a_{i3}) and (a_{j1}, a_{j2}, a_{j3}) be such row vectors. Then there must be an integer $l, 1 \leq l \leq m$ such that the l th row can not be the linear combination of the i th row and j th row. Whence, there are numbers λ, μ such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$.

Case 3 $\text{rank}A = 3, \text{rank}A^+ = 4$

In this case, there are three independent row vectors. Without loss of generality, let $(a_{i1}, a_{i2}, a_{i3}), (a_{j1}, a_{j2}, a_{j3})$ and (a_{k1}, a_{k2}, a_{k3}) be such row vectors. Then there must be an integer $l, 1 \leq l \leq m$ such that the l th row can not be the linear combination of the i th row, j th row and k th row. Thus there are numbers λ, μ, ν such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$. Combining the discussion of Case 1-Case 3, the proof is complete. \square

Notice that the linear equation system (*LEq3*) can be transformed to the following (*LEq3**) by elementary transformation, i.e., each j th row plus $-a_{j3}/a_{i3}$ times the i th row in (*LEq3*) for an integer $i, 1 \leq i \leq m$ with $a_{i3} \neq 0$,

$$A'X = (b'_1, b'_2, \dots, b'_m)^T \quad (\text{LEq3}^*)$$

with

$$A'^+ = \begin{bmatrix} a'_{11} & a'_{12} & 0 & b'_1 \\ \dots & \dots & \dots & \dots \\ a'_{(i-1)1} & a'_{(i-1)2} & 0 & b'_{i-1} \\ a_{i1} & a_{i2} & a_{i3} & b_i \\ a'_{(i+1)1} & a'_{(i+1)2} & 0 & b'_{i+1} \\ a'_{m1} & a'_{m2} & 0 & b'_m \end{bmatrix},$$

where $a'_{j1} = a_{j1} - a_{j3}a_{i1}/a_{i3}$, $a'_{j2} = a_{j2} - a_{j3}a_{i2}/a_{i3}$ and $b'_j = b_j - a_{j3}b_i/a_{i3}$ for integers $1 \leq j \leq m$. Applying Theorem 3.3, we get the a combinatorial characterizing on non-solvable linear systems (*LEq3*) following.

Theorem 4.2 *A linear equation system (LEq3) is non-solvable if and only if $G[LEq3] \not\simeq K_m$ or $G[LEq3^*] \simeq u + L_C(H)$, where H denotes a planar graph with order $|H| \geq 2$, size $m - 1$ and each edge a straight segment, $u + G$ the join of vertex u with G .*

Proof By Theorem 2.4, the linear equation system (LEq3) is non-solvable if and only if $G[LEq3] \not\simeq K_m$ or the linear equation system (LEq3*) is non-solvable, which implies that the linear equation subsystem following

$$BX' = (b'_1, \dots, b'_{i-1}, b'_{i+1}, \dots, b'_m)^T \quad (LEq2^*)$$

with

$$B = \begin{bmatrix} a'_{11} & a'_{12} \\ \dots & \dots \\ a'_{(i-1)1} & a'_{(i-1)2} \\ a'_{(i+1)1} & a'_{(i+1)2} \\ a'_{m1} & a'_{m2} \end{bmatrix} \quad \text{and} \quad X' = (x_1, x_2)^T$$

is non-solvable. Applying Theorem 3.3, we know that the linear equation subsystem (LEq2*) is non-solvable if and only if $G[LEq2^*] \simeq L_C(H)$, where H is a planar graph H of size $m - 1$ with each edge a straight segment. Thus the linear equation system (LEq3*) is non-solvable if and only if $G[LEq3^*] \simeq u + L_C(H)$. \square

§5. Linear Homeomorphisms Equations

A homeomorphism on \mathbb{R}^n is a continuous 1 - 1 mapping $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that its inverse h^{-1} is also continuous for an integer $n \geq 1$. There are indeed many such homeomorphisms on \mathbb{R}^n . For example, the linear transformations T on \mathbb{R}^n . A *linear homeomorphisms equation system* is such an equation system

$$AX = (b_1, b_2, \dots, b_m)^T \quad (L^hEq)$$

with $X = (h(x_1), h(x_2), \dots, h(x_n))^T$, where h is a homeomorphism and

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

for integers $m, n \geq 1$. Notice that the linear homeomorphism equation system

$$\begin{cases} a_{i1}h(x_1) + a_{i2}h(x_2) + \dots + a_{in}h(x_n) = b_i, \\ a_{j1}h(x_1) + a_{j2}h(x_2) + \dots + a_{jn}h(x_n) = b_j \end{cases}$$

is solvable if and only if the linear equation system

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \end{cases}$$

is solvable. Similarly, two linear homeomorphism equations are said *parallel* if they are non-solvable. Applying Theorems 2.6, 3.3, 4.2, we know the following result for linear homeomorphism equation systems (L^hEq).

Theorem 5.1 *Let (L^hEq) be a linear homeomorphism equation system for integers $m, n \geq 1$. Then*

- (1) $G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$ with $n_1 + n_2 + \dots + n_s = m$, where \mathcal{C}_i^h is the parallel family with $n_i = |\mathcal{C}_i^h|$ for integers $1 \leq i \leq s$ in (L^hEq) and (L^hEq) is non-solvable if $s \geq 2$;
- (2) If $n = 2$, (L^hEq) is non-solvable if and only if $G[L^hEq] \simeq L_C(H)$, where H is a planar graph of order $|H| \geq 2$ on \mathbb{R}^2 with each edge a homeomorphism of straight segment, and if $n = 3$, (L^hEq) is non-solvable if and only if $G[L^hEq] \not\simeq K_m$ or $G[LEq3^*] \simeq u + L_C(H)$, where H denotes a planar graph with order $|H| \geq 2$, size $m - 1$ and each edge a homeomorphism of straight segment.

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Roman Domination in Complementary Prism Graphs

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Abstract: A Roman domination function on a complementary prism graph GG^c is a function $f : V \cup V^c \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(GG^c)$ of a graph $G = (V, E)$ is the minimum of $\sum_{x \in V \cup V^c} f(x)$ over such functions, where the complementary prism GG^c of G is graph obtained from disjoint union of G and its complement G^c by adding edges of a perfect matching between corresponding vertices of G and G^c . In this paper, we have investigated few properties of $\gamma_R(GG^c)$ and its relation with other parameters are obtained.

Key Words: Graph, domination number, Roman domination number, Smarandachely Roman s-domination function, complementary prism, Roman domination of complementary prism.

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§1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G , respectively. For any vertex v of G , let $N(v)$ and $N[v]$ denote its open and closed neighborhoods respectively. $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a vertex (edge) cover of G . $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G . Let $\deg(v)$ be the degree of vertex v in G . Then $\Delta(G)$ and $\delta(G)$ be maximum and minimum degree of G , respectively. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M , then M is a perfect matching in G . The complement G^c of a graph G is the graph having the same set of vertices as G denoted by V^c and in which two vertices are adjacent, if and only if they are not adjacent in G . Refer to [5] for additional graph theory terminology.

A dominating set $D \subseteq V$ for a graph G is such that each $v \in V$ is either in D or adjacent to a vertex of D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . Further, a dominating set D is a minimal dominating set of G , if and only if for each vertex $v \in D$, $D - v$ is not a dominating set of G . For complete review on theory of domination

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and its related parameters, we refer [1], [6] and [7].

For a graph $G = (V, E)$, let $f : V \rightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f , where $V_i = \{v \in V / f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. There exist 1-1 correspondence between the functions $f : V \rightarrow \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V . Thus we write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ is a Roman dominating function (RDF) if $V_2 \succ V_0$, where \succ signifies that the set V_2 dominates the set V_0 . The weight of a Roman dominating function is the value $f(V) = \sum_{v \in V} f(v) = 2|V_2| + |V_1|$. Roman dominating number $\gamma_R(G)$, equals the minimum weight of an RDF of G , we say that a function $f = (V_0, V_1, V_2)$ is a γ_R -function if it is an RDF and $f(V) = \gamma_R(G)$. Generally, let $I \subset \{0, 1, 2, \dots, n\}$. A *Smarandachely Roman s -dominating function* for an integer s , $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2, \dots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s -dominating function is nothing but the Roman domination function. For more details on Roman dominations and its related parameters we refer [3]-[4] and [9]-[11].

In [8], Haynes et al., introduced the concept of domination and total domination in complementary prisms. Analogously, we initiate the Roman domination in complementary prism as follows:

A Roman domination function on a complementary prism graph GG^c is a function $f : V \cup V^c \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(GG^c)$ of a graph $G = (V, E)$ is the minimum of $\sum_{x \in V \cup V^c} f(x)$ over such functions, where the complementary prism GG^c of G is graph obtained from disjoint union of G and its complement G^c by adding edges of a perfect matching between corresponding vertices of G and G^c .

§2. Results

We begin by making a couple of observations.

Observation 2.1 For any graph G with order n and size m ,

$$m(GG^c) = n(n+1)/2.$$

Observation 2.2 For any graph G ,

- (i) $\beta_1(GG^c) = n$.
- (ii) $\alpha_1(GG^c) + \beta_1(GG^c) = 2n$.

Proof Let G be a graph and GG^c be its complementary prism graph with perfect matching M . If one to one correspondence between vertices of a graph G and its complement G^c in GG^c , then GG^c has even order and M is a 1-regular spanning sub graph of GG^c , thus (i) follows and due to the fact of $\alpha_1(G) + \beta_1(G) = n$, (ii) follows. \square

Observation 2.3 For any graph G ,

$$\gamma(GG^c) = n$$

if and only if G or G^c is totally disconnected graph.

Proof Let there be n vertices of degree 1 in GG^c . Let D be a dominating set of GG^c and v be a vertex of G of degree $n - 1$, $v \in D$. In GG^c , v dominates n vertices and remaining $n - 1$ vertices are pendent vertices which has to dominate itself. Hence $\gamma(GG^c) = n$. Conversely, if $\gamma(GG^c) = n$, then there are n vertices in minimal dominating set D . \square

Theorem 2.1 For any graph G ,

$$\gamma_R(GG^c) = \alpha_1(GG^c) + \beta_1(GG^c)$$

if and only if G being an isolated vertex.

Proof If G is an isolated vertex, then GG^c is K_2 and $\gamma_R(GG^c) = 2, \alpha_1(GG^c) = 1$ and $\beta_1(GG^c) = 1$. Conversely, if $\gamma_R(GG^c) = \alpha_1(GG^c) + \beta_1(GG^c)$. By above observation, then we have $\gamma_R(GG^c) = 2|V_2| + |V_1|$. Thus we consider the following cases:

Case 1 If $V_2 = \phi, |V_1| = 2$, then $V_0 = \phi$ and $GG^c \cong K_2$.

Case 2 If $|V_2| = 1, |V_1| = \phi$, then GG^c is a complete graph.

Hence the result follows. \square

Theorem 2.2 Let G and G^c be two complete graphs then GG^c is also complete if and only if $G \cong K_1$.

Proof If $G \cong K_1$ then $G^c \cong K_1$ and $GG^c \cong K_2$ which is a complete graph. Conversely, if GG^c is complete graph then any vertex v of G is adjacent to $n - 1$ vertices of G and n vertices of G^c . According to definition of complementary prism this is not possible for graph other than K_1 . \square

Theorem 2.3 For any graph G ,

$$\gamma(GG^c) < \gamma_R(GG^c) \leq 2\gamma(GG^c).$$

Further, the upper bound is attained if $V_1(GG^c) = \phi$.

Proof Let $f = (V_0, V_1, V_2)$ be γ_R -function. If $V_2 \succ V_0$ and $(V_1 \cup V_2)$ dominates GG^c , then $\gamma(GG^c) < |V_1 \cup V_2| = |V_1| + 2|V_2| = \gamma_R(GG^c)$. Thus the result follows.

Let $f = (V_0, V_1, V_2)$ be an RDF of GG^c with $|D| = \gamma(GG^c)$. Let $V_2 = D, V_1 = \phi$ and $V_0 = V - D$. Since f is an RDF and $\gamma_R(GG^c)$ denotes minimum weight of $f(V)$. It follows $\gamma_R(GG^c) \leq f(V) = |V_1| + 2|V_2| = 2|S| = 2\gamma(GG^c)$. Hence the upper bound follows. For graph GG^c , let v be vertex not in V_1 , implies that either $v \in V_2$ or $v \in V_0$. If $v \in V_2$ then $v \in D$, $\gamma_R(GG^c) = 2|V_2| + |V_1| = 2|D| = 2\gamma(GG^c)$. If $v \in V_0$ then $N(v) \subseteq V_2$ or $N(v) \subseteq V_0$ as v does not belong to V_1 . Hence the result. \square

Theorem 2.4 For any graph G ,

$$2 \leq \gamma_R(GG^c) \leq (n+1).$$

Further, the lower bound is attained if and only if $G \cong K_1$ and the upper bound is attained if G or G^c is totally disconnected graph.

Proof Let G be a graph with $n \geq 1$. If $f = \{V_0, V_1, V_2\}$ be a RDF of GG^c , then $\gamma_R(GG^c) \geq 2$. Thus the lower bound follows.

Upper bound is proved by using mathematical induction on number of vertices of G . For $n = 1$, $GG^c \cong K_2$, $\gamma_R(GG^c) = n + 1$. For $n = 2$, $GG^c \cong P_4$, $\gamma_R(GG^c) = n + 1$. Assume the result to be true for some graph H with $n - 1$ vertices, $\gamma_R(HH^c) \leq n$. Let G be a graph obtained by adding a vertex v to H . If v is adjacent to a vertex w in H which belongs to V_2 , then $v \in V_0$, $\gamma_R(GG^c) = n < n + 1$. If v is adjacent to a vertex either in V_0 or V_1 , then $\gamma_R(GG^c) = n + 1$. If v is adjacent to all vertices of H then $\gamma_R(GG^c) < n < n + 1$. Hence upper bound follows for any number of vertices of G .

Now, we prove the second part. If $G \cong K_1$, then $\gamma_R(GG^c) = 2$. On the other hand, if $\gamma_R(GG^c) = 2 = 2|V_2| + |V_1|$ then we have following cases:

Case 1 If $|V_2| = 1, |V_1| = 0$, then there exist a vertex $v \in V(GG^c)$ such that degree of $v = (n - 1)$, thus one and only graph with this property is $GG^c \cong K_2$. Hence $G = K_1$.

Case 2 If $|V_2| = 0, |V_1| = 2$, then there are only two vertices in the GG^c which are connected by an edge. Hence the result.

If G is totally disconnected then G^c is a complete graph. Any vertex v^c in G^c dominates n vertices in GG^c . Remaining $n - 1$ vertices of GG^c are in V_1 . Hence $\gamma_R(GG^c) = n + 1$. \square

Proposition 2.1([3]) For any path P_n and cycle C_n with $n \geq 3$ vertices,

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil,$$

where $\lceil x \rceil$ is the smallest integer not less than x .

Theorem 2.5 For any graph G ,

(i) if $G = P_n$ with $n \geq 3$ vertices, then

$$\gamma_R(GG^c) = 4 + \lceil 2(n-3)/3 \rceil;$$

(ii) if $G = C_n$ with $n \geq 4$ vertices, then

$$\gamma_R(GG^c) = 4 + \lceil 2(n-2)/3 \rceil.$$

Proof (i) Let $G = P_n$ be a path with $n \geq 3$ vertices. Then we have the following cases:

Case 1 Let $f = (V_0, V_1, V_2)$ be an RDF and a pendent vertex v is adjacent to a vertex u in G . The vertex v^c is not adjacent to a vertex u^c in V^c . But the vertex of v^c in V^c is adjacent

to n vertices of GG^c . Let $v^c \in V_2$ and $N(v^c) \subseteq V_0$. There are n vertices left and $u^c \in N[u]$ but $\{N(u^c) - u\} \subseteq V_0$. Hence $u \in V_2, N(u) \subseteq V_0$. There are $(n-3)$ vertices left, whose induced subgraph H forms a path with $\gamma_R(H) = \lceil 2(n-3)/3 \rceil$, this implies that $\gamma_R(G) = 4 + \lceil 2(n-3)/3 \rceil$.

Case 2 If v is not a pendent vertex, let it be adjacent to vertices u and w in G . Repeating same procedure as above case, $\gamma_R(GG^c) = 6 + \lceil 2(n-3)/3 \rceil$, which is a contradiction to fact of RDF.

(ii) Let $G = C_n$ be a cycle with $n \geq 4$ vertices. Let $f = (V_0, V_1, V_2)$ be an RDF and w be a vertex adjacent to vertex u and v in G , and w^c is not adjacent to u^c and v^c in V^c . But w^c is adjacent to $(n-2)$ vertices of GG^c . Let $w^c \in V_2$ and $N(w^c) \subseteq V_0$. There are $(n+1)$ -vertices left with u^c or $v^c \in V_2$. With out loss of generality, let $u^c \in V_2, N(u^c) \subseteq V_0$. There are $(n-2)$ vertices left, whose induced subgraph H forms a path with $\gamma_R(H) = \lceil 2(n-2)/3 \rceil$ and $V_2 = \{w, u^c\}$, this implies that $\gamma_R(G) = 4 + \lceil 2(n-2)/3 \rceil$. \square

Theorem 2.6 For any graph G ,

$$\max\{\gamma_R(G), \gamma_R(G^c)\} < \gamma_R(GG^c) \leq (\gamma_R(G) + \gamma_R(G^c)).$$

Further, the upper bound is attained if and only if the graph G is isomorphic with K_1 .

Proof Let G be a graph and let $f : V \rightarrow \{0, 1, 2\}$ and $f = (V_0, V_1, V_2)$ be RDF. Since GG^c has $2n$ vertices when G has n vertices, hence $\max\{\gamma_R(G), \gamma_R(G^c)\} < \gamma_R(GG^c)$ follows.

For any graph G with $n \geq 1$ vertices. By Theorem 2.4, we have $\gamma_R(GG^c) \leq (n+1)$ and $(\gamma_R(G) + \gamma_R(G^c)) \leq (n+2) = (n+1) + 1$. Hence the upper bound follows.

Let $G \cong K_1$. Then $GG^c = K_2$, thus the upper bound is attained. Conversely, suppose $G \not\cong K_1$. Let u and v be two adjacent vertices in G and u is adjacent to v and u^c in GG^c . The set $\{u, v^c\}$ is a dominating set out of which $u \in V_2, v^c \in V_1$. $\gamma_R(G) = 2, \gamma_R(G^c) = 0$ and $\gamma_R(GG^c) = 3$ which is a contradiction. Hence no two vertices are adjacent in G . \square

Theorem 2.7 If degree of every vertex of a graph G is one less than number of vertices of G , then

$$\gamma_R(GG^c) = \gamma(GG^c) + 1.$$

Proof Let $f = (V_0, V_1, V_2)$ be an RDF and let v be a vertex of G of degree $n-1$. In GG^c , v is adjacent to n vertices. If D is a minimum dominating set of GG^c then $v \in D$, $v \in V_2$ also $N(v) \subseteq V_0$. Remaining $n-1$ belongs to V_1 and D . $|D| = \gamma(GG^c) = n$ and $\gamma_R(GG^c) = n+1 = \gamma(GG^c) + 1$. \square

Theorem 2.8 For any graph G with $n \geq 1$ vertices,

$$\gamma_R(GG^c) \leq [2n - (\Delta(GG^c) + 1)].$$

Further, the bound is attained if G is a complete graph.

Proof Let G be any graph with $n \geq 1$ vertices. Then GG^c has $2n$ - vertices. Let $f = (V_0, V_1, V_2)$ be an RDF and v be any vertex of GG^c such that $\deg(v) = \Delta(GG^c)$. Then v

dominates $\Delta(GG^c) + 1$ vertices. Let $v \in V_2$ and $N(v) \subseteq V_0$. There are $(2n - (\Delta(GG^c) + 1))$ vertices left in GG^c , which belongs to one of V_0 , V_1 or V_2 . If all these vertices $\in V_1$, then $\gamma_R(GG^c) = 2|V_2| + |V_1| = 2 + (2n - \Delta(GG^c) + 1) = 2n - \Delta(GG^c) + 1$. Hence lower bound is attained when $G \cong K_n$, where v is a vertex of G . If not all remaining vertices belong to V_1 , then there may be vertices belonging to V_2 and which implies their neighbors belong to V_0 . Hence the result follows. \square

Theorem 2.9 For any graph G ,

$$\gamma_R(GG^c)^c \leq \gamma_R(GG^c).$$

Further, the bound is attained for one of the following conditions:

- (i) $GG^c \cong (GG^c)^c$;
- (ii) GG^c is a complete graph.

Proof Let G be a graph, GG^c be its complementary graph and $(GG^c)^c$ be complement of complementary prism. According to definition of GG^c there should be one to one matching between vertices of G and G^c , where as in $(GG^c)^c$ there will be one to $(n-1)$ matching between vertices of G and G^c implies that adjacency of vertices will be more in $(GG^c)^c$. Hence the result. If $GG^c \cong (GG^c)^c$, domination and Roman domination of these two graphs are same. The only complete graph GG^c can be is K_2 . $(GG^c)^c$ will be two isolated vertices, $\gamma_R(GG^c) = 2$ and $\gamma_R(GG^c)^c = 2$. Hence bound is attained. \square

To prove our next results, we make use of following definitions:

A *rooted tree* is a tree with a countable number of vertices, in which a particular vertex is distinguished from the others and called the root. In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root; every vertex except the root has a unique parent. A child of a vertex v is a vertex of which v is the parent. A leaf is a vertex without children.

A graph with exactly one induced cycle is called *unicyclic*.

Theorem 2.10 For any rooted tree T ,

$$\gamma_R(TT^c) = 2|S_2| + |S_1|,$$

where $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$.

Proof Let T be a rooted tree and $f = (V_0, V_1, V_2)$ be *RDF* of a complementary prism TT^c . We label all parent vertices of T as P_1, P_2, \dots, P_k where P_k is root of a tree T . Let S_p be set of all parent vertices of T , S_l be set of all leaf vertices of T and $v \in S_l$ be a vertex farthest from P_k . The vertex v^c is adjacent to $(n-1)$ -vertices in TT^c . Let $v^c \in S_2$, and $N(v^c) \subseteq V_0$. Let P_1 be parent vertex of $v \in T$. For $i=1$ to k if P_i is not assigned weight then $P_i \in S_2$ and $N(P_i) \subseteq V_0$. If P_i is assigned weight and check its leaf vertices in T , then we consider the following cases:

Case 1 If P_i has at least 2 leaf vertices, then $P_i \in S_2$ and $N(P_i) \subseteq V_0$.

Case 2 If P_i has at most 1 leaf vertex, then all such leaf vertices belong to S_1 . Thus $\gamma_R(GG^c) = 2|S_2| + |S_1|$ follows. \square

Theorem 2.11 *Let G^c be a complement of a graph G . Then the complementary prism GG^c is*

- (i) *isomorphic with a tree T if and only if G or G^c has at most two vertices.*
- (ii) *$(n+1)/2$ -regular graph if and only if G is $(n-1)/2$ -regular.*
- (iii) *unicyclic graph if and only if G has exactly 3 vertices.*

Proof (i) Suppose GG^c is a tree T with the graph G having minimum three vertices. Then we have the following cases:

Case 1 Let u, v and w be vertices of G with v is adjacent to both u and w . In GG^c , u^c is connected to u and w^c also v^c is connected to v . Hence there is a closed path $u-v-w-w^c-u^c-u$, which is a contradicting to our assumption.

Case 2 If vertices u, v and w are totally disconnected in G , then G^c is a complete graph. Since every complete graph G with $n \geq 3$ has cycle. Hence GG^c is not a tree.

Case 3 If u and v are adjacent but which is not adjacent to w in G , then in GG^c there is a closed path $u-u^c-w^c-v^c-v^c-u$, again which is a contradicting to assumption.

On the other hand, if G has one vertex, then $GG^c \cong K_2$ and if G have two vertices, then $GG^c \cong P_4$. In both the cases GG^c is a tree.

(ii) Let G be r -regular graph, where $r = (n-1)/2$, then G^c is $n-r-1$ regular. In GG^c , degree of every vertex in G is $r+1 = (n+1)/2$ and degree of every vertex in G^c is $n-r = (n+1)/2$, which implies GG^c is $(n+1)/2$ -regular. Conversely, suppose GG^c is $s = (n+1)/2$ -regular. Let E be set of all edges making perfect match between G and G^c . In $GG^c - E$, G is $s-1$ -regular and G^c is $(n-s-1)$ -regular. Hence the graph G is $(n-1)/2$ -regular.

(iii) If GG^c has at most two vertices, then from (i), GG^c is a tree. Minimum vertices required for a graph to be unicyclic is 3. Because of perfect matching in complementary prism and G and G^c are connected if there are more than 3 vertices there will be more than 1 cycle. \square

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Enumeration of Rooted Nearly 2-Regular Simple Planar Maps

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Abstract: This paper discusses the enumeration of rooted nearly 2-regular simple planar maps and presents some formulae for such maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters.

Key Words: Smarandachely map, simple map, nearly 2-regular map, enumerating function, functional equation, Lagrangian inversion, Lagrangian inversion.

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§1. Introduction

Let S be a surface. For an integer $k \geq 0$, a *Smarandachely k -map* on S is such a pseudo-map on S just with k faces that not being 2-cell. If $k = 0$, such a Smarandachely map is called *map*. In the field of enumerating planar maps, many functional equations for a variety of sets of planar maps have been found and some solutions of the equations are obtained. Some nice skills are applied in this area and they have set up the foundation of enumerative theory [2], [5], [6] and [9–13]. But the discussion on enumerating function of simple planar maps is very few. All the results obtained so far are almost concentrated in general simple planar maps [3], [4], [7] and [8]. In 1997, Cai [1] investigated for the first time the enumeration of simple Eulerian planar maps with the valency of root-vertex, the number of inner edges and the valency of root-face as parameters and a functional equation satisfied by its enumerating function was obtained, but it is very complicated and its solution has not been found up to now.

In this paper we treat the enumeration of rooted nearly 2-regular simple planar maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters. Several explicit expressions of its enumerating functions are obtained and one of them is summation-free.

Now, we define some basic concepts and terms. In general, rooting a map means distinguishing one edge on the boundary of the outer face as the root-edge, and one end of that edge as the root-vertex. In diagrams we usually represent the root-edge as an edge with an arrow on

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the outer face, the arrow being drawn from the root-vertex to the other end. So the outer face is also called the root-face. A planar map with a rooting is said to be a rooted planar map. We say that two rooted planar maps are combinatorially equivalent or up to root-preserving isomorphism if they are related by 1-1 correspondence of their elements, which maps vertices onto vertices, edges onto edges and faces onto faces, which preserves incidence relations and which preserves the root-vertex, root-edge and root-face. Otherwise, combinatorially inequivalent or nonisomorphic here.

A nearly 2-regular map is a rooted map such that all vertices probably except the root-vertex are of valency 2. A map is said to be simple, if there is neither loop nor parallel edge.

For a set of some maps \mathcal{M} , the enumerating function discussed in this paper is defined as

$$f_{\mathcal{M}}(x, y, z) = \sum_{M \in \mathcal{M}} x^{l(M)} y^{p(M)} z^{q(M)}, \quad (1)$$

where $l(M)$, $p(M)$ and $q(M)$ are the root-face valency, the number of nonrooted vertices and the number of inner faces of M , respectively.

Furthermore, we introduce some other enumerating functions for \mathcal{M} as follows:

$$\begin{aligned} g_{\mathcal{M}}(x, y) &= \sum_{M \in \mathcal{M}} x^{l(M)} y^{n(M)}, \\ h_{\mathcal{M}}(y, z) &= \sum_{M \in \mathcal{M}} y^{p(M)} z^{q(M)}, \\ H_{\mathcal{M}}(y) &= \sum_{M \in \mathcal{M}} y^{n(M)}, \end{aligned} \quad (2)$$

where $l(M)$, $p(M)$ and $q(M)$ are the same in (1) and $n(M)$ is the number of edges of M , that is

$$\begin{aligned} g_{\mathcal{M}}(x, y) &= f_{\mathcal{M}}(x, y, y), \quad h_{\mathcal{M}}(y, z) = f_{\mathcal{M}}(1, y, z), \\ H_{\mathcal{M}}(y) &= g_{\mathcal{M}}(1, y) = h_{\mathcal{M}}(y, y) = f_{\mathcal{M}}(1, y, y). \end{aligned} \quad (3)$$

For the power series $f(x)$, $f(x, y)$ and $f(x, y, z)$, we employ the following notations:

$$\partial_x^n f(x), \quad \partial_{(x,y)}^{(m,p)} f(x, y) \quad \text{and} \quad \partial_{(x,y,z)}^{(m,p,q)} f(x, y, z)$$

to represent the coefficients of x^m in $f(x)$, $x^m y^p$ in $f(x, y)$ and $x^m y^p z^q$ in $f(x, y, z)$, respectively. Terminologies and notations not explained here can be found in [11].

§2. Functional Equations

In this section we will set up the functional equations satisfied by the enumerating functions for rooted nearly 2-regular simple planar maps.

Let \mathcal{E} be the set of all rooted nearly 2-regular simple planar maps with convention that the vertex map ϑ is in \mathcal{E} for convenience. From the definition of a nearly 2-regular simple map, for any $M \in \mathcal{E} - \vartheta$, each edge of M is contained in only one circuit. The circuit containing the root-edge is called the root circuit of M , and denoted by $C(M)$.

It is clear that the length of the root circuit is no more than the root-face valency, and

$$\mathcal{E} = \mathcal{E}_0 + \bigcup_{i \geq 3} \mathcal{E}_i, \quad (4)$$

where

$$\mathcal{E}_i = \{M \mid M \in \mathcal{E}, \text{ the length of } C(M) \text{ is } i\} \quad (5)$$

and \mathcal{E}_0 is only consist of the vertex map ϑ .

It is easy to see that the enumerating function of \mathcal{E}_0 is

$$f_{\mathcal{E}_0}(x, y, z) = 1. \quad (6)$$

For any $M \in \mathcal{E}_i (i \geq 3)$, the root circuit divides $M - C(M)$ into two domains, the inner domain and outer domain. The submap of M in the outer domain is a general map in \mathcal{E} , while the submap of M in the inner domain does not contribute the valency of the root-face of M . Therefore, the enumerating function of \mathcal{E}_i is

$$f_{\mathcal{E}_i}(x, y, z) = x^i y^{i-1} z h f, \quad (7)$$

where $h = h_{\mathcal{E}}(y, z) = f_{\mathcal{E}}(1, y, z)$.

Theorem 2.1 *The enumerating function $f = f_{\mathcal{E}}(x, y, z)$ satisfies the following equation:*

$$f = \left[1 - \frac{x^3 y^2 z h}{1 - x y} \right]^{-1}, \quad (8)$$

where $h = h_{\mathcal{E}}(y, z) = f_{\mathcal{E}}(1, y, z)$.

Proof By (4), (6) and (7), we have

$$\begin{aligned} f &= 1 + \sum_{i \geq 3} x^i y^{i-1} z h f \\ &= 1 + \frac{x^3 y^2 z h f}{1 - x y}, \end{aligned}$$

which is equivalent to the theorem. □

Let $y = z$ in (8). Then we have

Corollary 2.1 *The enumerating function $g = g_{\mathcal{E}}(x, y)$ satisfies the following equation:*

$$g = \left[1 - \frac{x^3 y^3 H}{1 - x y} \right]^{-1}, \quad (9)$$

where $H = H_{\mathcal{E}}(y) = g_{\mathcal{E}}(1, y)$.

Let $x = 1$ in (8). Then we obtain

Corollary 2.2 *The enumerating function $h = h_{\mathcal{E}}(y, z)$ satisfies the following equation:*

$$y^2 z h^2 - (1 - y)h - y + 1 = 0. \quad (10)$$

Further, let $y = z$ in (10). Then we have

Corollary 2.3 *The enumerating function $H = H_{\mathcal{E}}(y)$ satisfies the following equation:*

$$y^3 H^2 - (1 - y)H - y + 1 = 0. \quad (11)$$

§3. Enumeration

In this section we will find the explicit formulae for enumerating functions $f = f_{\mathcal{E}}(x, y, z)$, $g = g_{\mathcal{E}}(x, y)$, $h = h_{\mathcal{E}}(y, z)$ and $H = H_{\mathcal{E}}(y)$ by using Lagrangian inversion.

By (10) we have

$$h = \frac{(1 - y) \left(1 - \sqrt{1 - \frac{4y^2 z}{1 - y}} \right)}{2y^2 z}. \quad (12)$$

Let

$$y = \frac{\theta}{1 + \theta}, \quad yz = \eta(1 - \theta\eta). \quad (13)$$

By substituting (13) into (12), one may find that

$$h = \frac{1}{1 - \theta\eta}. \quad (14)$$

By (13) and (14), we have the following parametric expression of $h = h_{\mathcal{E}}(y, z)$:

$$\begin{aligned} y &= \frac{\theta}{1 + \theta}, \quad yz = \eta(1 - \theta\eta), \\ h &= \frac{1}{1 - \theta\eta} \end{aligned} \quad (15)$$

and from which we get

$$\Delta_{(\theta, \eta)} = \begin{vmatrix} \frac{1}{1 + \theta} & 0 \\ * & \frac{1 - 2\theta\eta}{1 - \theta\eta} \end{vmatrix} = \frac{1 - 2\theta\eta}{(1 + \theta)(1 - \theta\eta)}. \quad (16)$$

Theorem 3.1 *The enumerating function $h = h_{\mathcal{E}}(y, z)$ has the following explicit expression:*

$$h_{\mathcal{E}}(y, z) = 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(2q)!(p - q - 1)!}{q!(q + 1)!(p - 2q)!(q - 1)!} y^p z^q. \quad (17)$$

Proof By employing Lagrangian inversion with two parameters, from (15) and (16) one

may find that

$$\begin{aligned}
h_{\mathcal{E}}(y, z) &= \sum_{p, q \geq 0} \partial_{(\theta, \eta)}^{(p, q)} \frac{(1 + \theta)^{p-1} (1 - 2\theta\eta)}{(1 - \theta\eta)^{q+2}} y^{p+q} z^q \\
&= \sum_{p \geq 0} \sum_{q=0}^p \partial_{(\theta, \eta)}^{(p-q, q)} \frac{(1 + \theta)^{p-q-1} (1 - 2\theta\eta)}{(1 - \theta\eta)^{q+2}} y^p z^q \\
&= 1 + \sum_{p \geq 1} \sum_{q=1}^p \left[\partial_{(\theta, \eta)}^{(p-q, q)} \frac{(1 + \theta)^{p-q-1}}{(1 - \theta\eta)^{q+2}} - 2 \partial_{(\theta, \eta)}^{(p-q-1, q-1)} \frac{(1 + \theta)^{p-q-1}}{(1 - \theta\eta)^{q+2}} \right] y^p z^q \\
&= 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(2q)!}{q!(q+1)!} \partial_{\theta}^{p-2q} (1 + \theta)^{p-q-1} y^p z^q \\
&= 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(2q)!(p-q-1)!}{q!(q+1)!(p-2q)!(q-1)!} y^p z^q,
\end{aligned}$$

which is just the theorem. \square

In what follows we present a corollary of Theorem 3.1.

Corollary 3.1 *The enumerating function $H = H_{\mathcal{E}}(y)$ has the following explicit expression:*

$$H_{\mathcal{E}}(y) = 1 + \sum_{n \geq 3} \sum_{q=1}^{\lfloor \frac{n}{3} \rfloor} \frac{(2q)!(n-2q-1)!}{q!(q+1)!(n-3q)!(q-1)!} y^n. \quad (18)$$

Proof It follows immediately from (17) by putting $x = y$ and $n = p + q$. \square

Now, let

$$x = \frac{\xi(1 + \theta)}{1 + \xi\theta}. \quad (19)$$

By substituting (15) and (19) into Equ. (8), one may find that

$$f = \frac{1}{1 - \frac{\xi^3 \theta \eta (1 + \theta)^2}{(1 + \xi\theta)^2}}. \quad (20)$$

By (15), (19) and (20), we have the parametric expression of the function $f = f_{\mathcal{E}}(x, y, z)$ as follows:

$$\begin{aligned}
x &= \frac{\xi(1 + \theta)}{1 + \xi\theta}, \quad y = \frac{\theta}{1 + \theta}, \\
yz &= \eta(1 - \theta\eta), \quad f = \frac{1}{1 - \frac{\xi^3 \theta \eta (1 + \theta)^2}{(1 + \xi\theta)^2}}.
\end{aligned} \quad (21)$$

According to (21), we have

$$\Delta_{(\xi, \theta, \eta)} = \begin{vmatrix} \frac{1}{1 + \xi\theta} & * & 0 \\ 0 & \frac{1}{1 + \theta} & 0 \\ 0 & * & \frac{1 - 2\theta\eta}{1 - \theta\eta} \end{vmatrix} = \frac{1 - 2\theta\eta}{(1 + \xi\theta)(1 + \theta)(1 - \theta\eta)}. \quad (22)$$

Theorem 3.2 *The enumerating function $f = f_{\mathcal{E}}(x, y, z)$ has the following explicit expression:*

$$f_{\mathcal{E}}(x, y, z) = 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{l=3}^{p+q} \sum_{k=\max\{1, \lceil \frac{l+2q-p}{3} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \frac{(2q-k-1)!k}{(q-k)!q!} \binom{l-2k-1}{l-3k} \\ \times \binom{p-q-l+2k-1}{p-2q-l+3k} x^l y^p z^q. \quad (23)$$

Proof By using Lagrangian inversion with three variables, from (21) and (22) one may find that

$$\begin{aligned} f_{\mathcal{E}}(x, y, z) &= \sum_{l, p, q \geq 0} \partial_{(\xi, \theta, \eta)}^{(l, p, q)} \frac{(1 + \xi\theta)^{l-1} (1 + \theta)^{p-l-1} (1 - 2\theta\eta)}{(1 - \theta\eta)^{q+1} \left[1 - \frac{\xi^3 \theta \eta (1 + \theta)^2}{(1 + \xi\theta)^2} \right]} x^l y^p z^q \\ &= \sum_{l, p \geq 0} \sum_{q=0}^p \partial_{(\xi, \theta, \eta)}^{(l, p-q, q)} \frac{(1 + \xi\theta)^{l-1} (1 + \theta)^{p-q-l-1} (1 - 2\theta\eta)}{(1 - \theta\eta)^{q+1} \left[1 - \frac{\xi^3 \theta \eta (1 + \theta)^2}{(1 + \xi\theta)^2} \right]} x^l y^p z^q \\ &= 1 + \sum_{l, p \geq 1} \sum_{q=1}^p \sum_{k=0}^{\min\{\lfloor \frac{l}{3} \rfloor, p-q, q\}} \partial_{(\xi, \theta, \eta)}^{(l-3k, p-q-k, q-k)} \frac{(1 + \xi\theta)^{l-2k-1}}{(1 - \theta\eta)^{q+1}} \\ &\quad \times (1 + \theta)^{p-q-l+2k-1} (1 - 2\theta\eta) x^l y^p z^q \\ &= 1 + \sum_{l, p \geq 1} \sum_{q=1}^p \sum_{k=\max\{0, \lceil \frac{l+q-p}{2} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \binom{l-2k-1}{l-3k} \\ &\quad \times \partial_{(\theta, \eta)}^{(p-q-l+2k, q-k)} \frac{(1 + \theta)^{p-q-l+2k-1} (1 - 2\theta\eta)}{(1 - \theta\eta)^{q+1}} x^l y^p z^q \\ &= 1 + \sum_{l, p \geq 1} \sum_{q=1}^p \sum_{k=\max\{0, \lceil \frac{l+q-p}{2} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \binom{l-2k-1}{l-3k} \\ &\quad \times \left[\partial_{(\theta, \eta)}^{(p-q-l+2k, q-k)} \frac{(1 + \theta)^{p-q-l+2k-1}}{(1 - \theta\eta)^{q+1}} \right. \\ &\quad \left. - 2\partial_{(\theta, \eta)}^{(p-q-l+2k-1, q-k-1)} \frac{(1 + \theta)^{p-q-l+2k-1}}{(1 - \theta\eta)^{q+1}} \right] x^l y^p z^q \\ &= 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{l=3}^{p+q} \sum_{k=\max\{1, \lceil \frac{l+2q-p}{3} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \frac{(2q-k-1)!k}{(q-k)!q!} \binom{l-2k-1}{l-3k} \\ &\quad \times \partial_{\theta}^{p-2q-l+3k} (1 + \theta)^{p-q-l+2k-1} x^l y^p z^q \\ &= 1 + \sum_{p \geq 2} \sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{l=3}^{p+q} \sum_{k=\max\{1, \lceil \frac{l+2q-p}{3} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \frac{(2q-k-1)!k}{(q-k)!q!} \binom{l-2k-1}{l-3k} \\ &\quad \times \binom{p-q-l+2k-1}{p-2q-l+3k} x^l y^p z^q, \end{aligned}$$

which is what we wanted. \square

Finally, we give a corollary of Theorem 3.2.

Corollary 3.2 *The enumerating function $g = g_{\mathcal{E}}(x, y)$ has the following explicit expression:*

$$g_{\mathcal{E}}(x, y) = 1 + \sum_{n \geq 3} \sum_{l=3}^n \sum_{q=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{k=\max\{1, \lceil \frac{l+3q-n}{3} \rceil\}}^{\min\{\lfloor \frac{l}{3} \rfloor, q\}} \frac{(2q-k-1)!k}{(q-k)!q!} \binom{l-2k-1}{l-3k} \\ \times \binom{n-2q-l+2k-1}{n-3q-l+3k} x^l y^n. \quad (24)$$

Proof It follows soon from (23) by putting $x = y$ and $n = p + q$. \square

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On Pathos Total Semitotal and Entire Total Block Graph of a Tree

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Abstract: In this communication, the concept of pathos total semitotal and entire total block graph of a tree is introduced. Its study is concentrated only on trees. We present a characterization of graphs whose pathos total semitotal block graphs are planar, maximal outerplanar, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian and hamiltonian. Also, we present a characterization of those graphs whose pathos entire total block graphs are planar, maximal outerplanar, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian, hamiltonian and graphs with crossing number one.

Key Words: Pathos, path number, Smarandachely block graph, semitotal block graph, total block graph, pathos total semitotal block graph, pathos entire total block graph, pathos length, pathos point, inner point number.

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§1. Introduction

The concept of pathos of a graph G was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is G . The path number of a graph G is the number of paths in a pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [5]. For a graph $G(p, q)$ if $B = u_1, u_2, u_3, \dots, u_r; r \geq 2$ is a block of G . Then we say that point u_1 and block B are incident with each other, as are u_2 and B and soon. If two distinct blocks B_1 and B_2 are incident with a common cut point, then they are called adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph* $T_S^V(G)$ for a subset $V \subset V(G)$ is such a graph with vertices $V \cup \mathcal{B}$ in which two points are adjacent if and only if the corresponding members of G are adjacent in $\langle V \rangle_G$ or incident in G , where \mathcal{B} is the set of blocks of G . The semitotal block graph of a graph G denoted $T_b(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if

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members of G are incident. The total block graph of a graph G denoted by $T_B(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if the corresponding members of G are adjacent or incident. Also, a new concept called pathos semitotal and total block graph of a tree has been introduced by Muddebihal [10]. The pathos semitotal graph of a tree T denoted by $P_{T_b}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are incident and the lines lie on the corresponding path P_i of pathos. The pathos total block graph of a tree T denoted by $P_{T_B}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are adjacent or incident and the lines lie on the corresponding path P_i of pathos. Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines. The pathos total semitotal block graph of a tree T denoted by is defined as the graph whose point set is the union of set of points and set of blocks of T and the path of pathos of T in which two points are adjacent if and only if the corresponding members of T are incident and the lines lie on the corresponding path P_i of pathos. The pathos entire total block graph of a tree denoted by is defined as the graph whose set of points is the union of set of points, set of blocks and the path of pathos of T in which two points are adjacent if and only if the corresponding members of T are adjacent or incident and the lines lie on the corresponding path P_i of pathos. Since the system of pathos for T is not unique, the corresponding pathos total semitotal block graph and pathos entire total block graphs are also not unique.

In Figure 1, a tree T and its semi total block graph $T_b(T)$ and their pathos total semitotal block graph are shown. In Figure 2, a tree T and its total block graph $T_B(T)$ and their pathos entire total block graphs are shown.

The line degree of a line uv in T , pathos length in T , pathos point in T was defined by Muddebihal [9]. If G is planar, the inner point number $i(G)$ of G is the minimum number of points not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G) = 1$ as was given by Kulli [4].

We need the following results for our further results.

Theorem A([10]) *For any non-trivial tree T , the pathos semitotal block graph $P_{T_b}(T)$ of a tree T , whose points have degree d_i , then the number of points are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$, where k is the path number.*

Theorem B([10]) *For any non-trivial tree whose points have degree d_i , the number of points and lines in total block graph $T_B(T)$ of a tree T are $(2q + 1)$ and $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$.*

Theorem C([10]) *For any non-trivial tree T , the pathos total block graph $P_{T_B}(T)$ of a tree*

T , whose points have degree d_i , then the number of points in $P_{TB}(T)$ are $(2q + k + 1)$ and the number of lines are $\left(q + 2 + \sum_{i=1}^p d_i^2\right)$, where k is the path number.

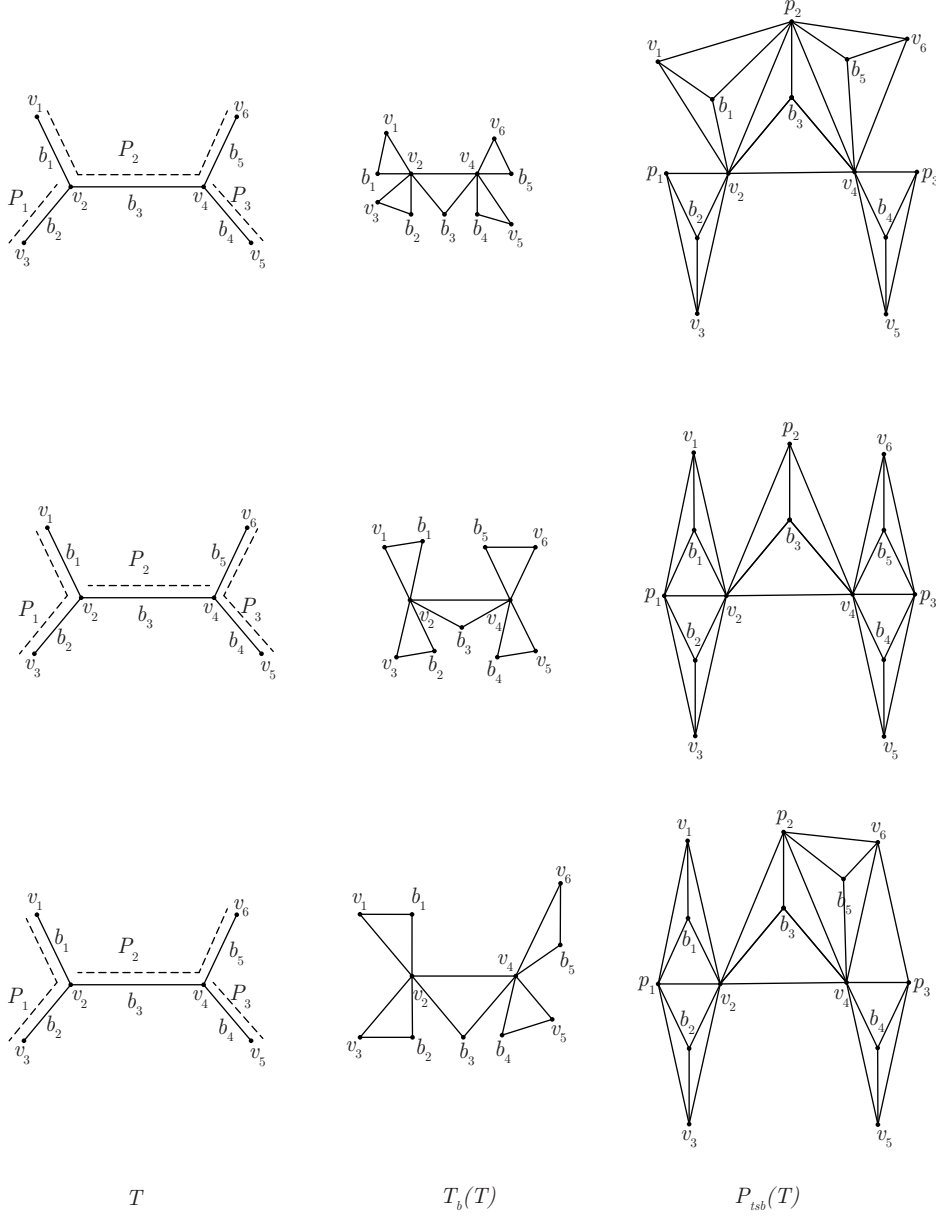


Figure 1

Theorem D ([7]) *The total block graph $T_B(G)$ of a graph G is planar if and only if G is outerplanar and every cut point of G lies on at most three blocks.*

Theorem E ([6]) *The total block graph $T_B(G)$ of a connected graph G is minimally nonouter-*

planar if and only if,

- (1) G if a cycle, or
- (2) G is a path of length $n \geq 2$, together with a point which is adjacent to any two adjacent points of P .

Theorem $F([8])$ The total block graph $T_B(G)$ of a graph G has crossing number one if and only if,

- (1) G is outerplanar and every cut point in G lies on at most 4 blocks and G has a unique cut point which lies on 4 blocks, or
- (2) G is minimally nonouterplanar, every cut point of G lies on at most 3 blocks and exactly one block of G is theta-minimally nonouterplanar.

Corollary $A([1])$ Every non-trivial tree T contains at least two end points.

§2. Pathos Total Semitotal Block Graph of a Tree

We start with a few preliminary results.

Remark 2.1 The number of blocks in pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is equal to the number of pathos in T .

Remark 2.2 If the pathos length of the path P_i of pathos in T is n , then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

In the following theorem we obtain the number of points and lines in pathos total semitotal block graph $P_{tsb}(T)$ of a tree T .

Theorem 2.1 For any non-trivial tree T , the pathos total semi total block graph $P_{tsb}(T)$ of a tree T , whose points have degree d_i , then the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$ and the number of lines are

$$\left(3q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2 \right)$$

where k is the path number.

Proof By Theorem A, the number of points in $P_{T_b}(T)$ are $(2q + k + 1)$, and by definition of $P_{tsb}(T)$, the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$, where k is the path number. Also by Theorem A, the number of lines in $P_{T_b}(T)$ are $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2 \right)$. The number of lines in $P_{tsb}(T)$ is equal to the sum of lines in $P_{T_b}(T)$ and the number of lines which lie on the lines (or blocks) of pathos, which are equal to q , since the number of lines are equal to the number of blocks in a tree T . Thus the number of lines in $P_{tsb}(T)$ is equal to

$$\left[q + (2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2) \right] = 3q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2. \quad \square$$

§3. Planar Pathos Total Semitotal Block Graphs

A criterion for pathos total semitotal block graph $P_{tsb}(T)$ of a tree T to be planar is presented in our next theorem.

Theorem 3.1 *For any non-trivial tree T , the pathos total semi total block graph $P_{tsb}(T)$ of a tree T is planar.*

Proof Let T be a non-trivial tree, then in $T_b(T)$ each block is a triangle. We have the following cases.

Case 1 Suppose G is a path, $G = P_n : u_1, u_2, u_3, \dots, u_n, n > 1$. Further, $V[T_b(T)] = \{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$, where $b_1, b_2, b_3, \dots, b_{n-1}$ are the corresponding block points. In pathos total semi total block graph $P_{tsb}(T)$ of a tree T , the pathos point w is adjacent to, $\{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$. For the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T , $\{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \dots, u_{n-1}b_{n-1}u_nw\} \in V[P_{tsb}(T)]$, in which each set $\{u_{n-1}b_{n-1}u_nw\}$ forms an induced subgraph as K_4 . Hence one can easily verify that each induced subgraphs of corresponding set $\{u_{n-1}b_{n-1}u_nw\}$ is planar. Hence $P_{tsb}(T)$ is planar.

Case 2 Suppose G is not a path. Then $V[T_b(G)] = \{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$ and $w_1, w_2, w_3, \dots, w_k$ be the pathos points. Since $u_{n-1}u_n$ is a line and $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$. Then in $P_{tsb}(G)$ the set $\{u_{n-1}, b_{n-1}, u_n, w\} \forall n > 1$, forms K_4 as an induced subgraphs. Hence $P_{tsb}(T)$ is planar. \square

The next theorem gives a minimally nonouterplanar $P_{tsb}(T)$.

Theorem 3.2 *For any non-trivial tree T , the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is minimally nonouterplanar if and only if $T = K_2$.*

Proof Suppose $T = K_3$, and $P_{tsb}(T)$ is minimally nonouterplanar, then $T_b(T) = K_4$ and one can easily verify that $i(P_{tsb}(T)) > 1$, a contradiction.

Suppose $T \neq K_2$. Now assume $T = K_{1,2}$ and $P_{tsb}(T)$ is minimally nonouterplanar. Then $T_b(T) = k_3 \cdot k_3$. Since $K_{1,2}$ has exactly one pathos and let v be a pathos point which is adjacent to all the points of $k_3 \cdot k_3$ in $P_{tsb}(T)$. Then one can easily see that, $i(P_{tsb}(T)) > 1$ a contradiction.

For converse, suppose $T = K_2$, then $T_b(T) = K_3$ and $P_{tsb}(T) = K_4$. Hence $P_{tsb}(T)$ is minimally nonouterplanar. \square

From Theorem 3.2, we developed the inner point number of a tree as shown in the following corollary.

Corollary 3.1 *For any non-trivial tree T with q lines, $i(P_{tsb}(T)) = q$.*

Proof The result is obvious for a tree with $q = 1$ and 2. Next we show that the result is true for $q \geq 3$. Now we consider the following cases.

Case 1 Suppose T is a path, $P : v_1, v_2, \dots, v_n$ such that $v_1v_2 = e_1, v_2v_3 = e_2 \dots, v_{n-1}v_n =$

e_{n-1} be the lines of P . Since each $e_i, 1 \leq i \leq n-1$, be a block of P , then in $T_b(P)$, each e_i is a point such that $V[T_b(P)] = V(P) \cup E(P)$. In $T_b(P)$ each $v_1e_1v_2, v_3e_2v_3, \dots, v_{n-1}e_{n-1}v_n$ forms a block in which each block is k_3 . Since each line is a block in P , then the number of k_3 's in $T_b(P)$ is equal to the numbers of lines in P . In $P_{tsb}(P)$, it has exactly one pathos. Then $V[P_{tsb}(P)] = V[T_b(P)] \cup \{P\}$ and P together with each block of $T_b(P)$ forms a block as $P_{tsb}(P)$. Now the points p, v_1, e_1, v_2 forms k_4 as a subgraph of a block $P_{tsb}(P)$. Similarly each $\{v_2, e_2, v_3, p\}, \{v_3, e_3, v_4, p\}, \dots, \{v_{n-1}, e_{n-1}, v_n, p\}$ forms k_4 as a subgraph of a block $P_{tsb}(P)$. One can easily find that each point $e_i, 1 \leq i \leq n-1$ lie in the interior region of k_4 , which implies that $i(P_{tsb}(P)) = q$.

Case 2 Suppose T is not a path, then T has at least one point of degree greater than two. Now assume T has exactly one point v , $\deg v \geq 3$. Then $T = K_{1,n}$. If $P_{tsb}(T)$ has inner point number two which is equal to $n = q$. Similarly if n is odd then $P_{tsb}(T)$ has $n-1$ blocks with inner point number two and exactly one block which is isomorphic to k_4 . Hence $i[P_{tsb}(K_{1,n})] = q$. Further this argument can be extended to a tree with at least two or more points of degree greater two. In each case we have $i[P_{tsb}(T)] = q$. \square

In the next theorem, we characterize the noneulerian $P_{tsb}(T)$.

Theorem 3.3 *For any non-trivial tree T , the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is noneulerian.*

Proof We have the following cases.

Case 1 Suppose $\Delta(T) \leq 2$ and if $p = 2$ points, then $P_{tsb}(T) = K_4$, which is noneulerian. If T is a path with $p > 2$ points. Then in $T_b(T)$ each block is a triangle such that they are in sequence with the vertices of $T_b(T)$ as $\{v_1, b_1, v_2, v_1\}$ an induced subgraph as a triangle in $T_b(T)$. Further $\{v_2, b_2, v_3, v_2\}, \{v_3, b_3, v_4, v_3\}, \dots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$, in which each set form a triangle as an induced subgraph of $T_b(T)$. Clearly one can easily verify that $T_b(T)$ is eulerian. Now this path has exactly one pathos point say k_1 , which is adjacent to $v_1, v_2, v_3, \dots, v_n, b_1, b_2, b_3, \dots, b_{n-1}$ in $P_{tsb}(T)$ in which all the points $v_1, v_2, v_3, \dots, v_n, b_1, b_2, b_3, \dots, b_{n-1} \in P_{tsb}(T)$ are of odd degree. Hence $P_{tsb}(T)$ is noneulerian.

Case 2 Suppose $\Delta(T) \geq 3$. Assume T has a unique point of degree ≥ 3 and also assume that $T = K_{1,n}$. Then in $T_b(T)$ each block is a triangle, such that there are n number of blocks which are K_3 with a common cut point k . Since the degree of a vertex $k = 2n$. One can easily verify that $T_b(K_{1,3})$ is eulerian. To form $P_{tsb}(T)$, $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives points of odd degree in $P_{tsb}(T)$. Hence $P_{tsb}(T)$ is noneulerian. \square

In the next theorem we characterize the hamiltonian $P_{tsb}(T)$.

Theorem 3.4 *For any non-trivial tree T , the pathos semitotal block graph $P_{tsb}(T)$ of a tree T is hamiltonian if and only if T is a path.*

Proof For the necessity, suppose T is a path and has exactly one path of pathos.

Let $V[T_b(T)] = \{u_1, u_2, u_3, \dots, u_n\} \{b_1, b_2, b_3, \dots, b_{n-1}\}$, where $b_1, b_2, b_3, \dots, b_{n-1}$ are

block points of T . Since each block is a triangle and each block consists of points as $B_1 = \{u_1, b_1, u_2\}, B_2 = \{u_2, b_2, u_3\}, \dots, B_m = \{u_m, b_m, u_{m+1}\}$. In $P_{tsb}(T)$ the pathos point w is adjacent to $\{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$. Hence $V[P_{tsb}(T)] = \{u_1, u_2, u_3, \dots, u_n\} \cup \{b_1, b_2, b_3, \dots, b_{n-1}\} \cup w$ form a spanning cycle as $w, u_1, b_1, u_2, b_2, u_3, \dots, u_{n-1}, b_{n-1}, u_n, w$ of $P_{tsb}(T)$. Clearly $P_{tsb}(T)$ is hamiltonian. Thus the necessity is proved.

For the sufficiency, suppose $P_{tsb}(T)$ is hamiltonian. Now we consider the following cases.

Case 1 Assume T is a path. Then T has at least one point with $\deg v \geq 3, \forall v \in V(T)$, suppose T has exactly one point u such that degree $u > 2$ and assume $G = T = K_{1,n}$. Now we consider the following subcases of case 1.

Subcase 1.1 For $K_{1,n}, n > 2$ and if n is even, then in $T_b(T)$ each block is k_3 . The number of path of pathos are $\frac{n}{2}$. Since n is even we get $\frac{n}{2}$ blocks in $P_{tsb}(T)$, each block contains two times of $\langle K_4 \rangle$ with some edges common. Since $P_{tsb}(T)$ has a cut points, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

Subcase 1.2 For $K_{1,n}, n > 2$ and n is odd, then the number of path of pathos are $\frac{n+1}{2}$, since n is odd we get $\frac{n-1}{2} + 1$ blocks in which $\frac{n-1}{2}$ blocks contains two times of $\langle K_4 \rangle$ which is nonline disjoint subgraph of $P_{tsb}(T)$ and remaining blocks is $\langle K_4 \rangle$. Since $P_{tsb}(T)$ contain a cut point, clearly $P_{tsb}(T)$ does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition. \square

§4. Pathos Entire Total Block Graph of a Tree

A tree T , its total block graph $T_B(T)$, and their pathos entire total block graphs $P_{etb}(T)$ are shown in Figure 2. We start with a few preliminary results.

Remark 4.1 If the pathos length of path P_i of pathos in T is n , then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

Remark 4.2 For any nontrivial tree T , the pathos entire total block graph $P_{etb}(T)$ is a block.

Theorem 4.1 For any non-trivial tree T , the pathos total block graph $P_{etb}(T)$ of a tree T , whose points have degree d_i , then the number of points in $P_{etb}(T)$ are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \sum_{i=1}^p d_i^2\right)$, where k is the path number.

Proof By Theorem C, the number of points in $P_{T_B}(T)$ are $(2q + k + 1)$, by definition of $P_{etb}(T)$, the number of points in $P_{etb}(T)$ are $(2q + k + 1)$, where k is the path number in T . Also by Theorem B, the number of lines in $T_B(T)$ are $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$. By Theorem C, The number of lines in $P_{T_B}(T)$ are $\left(q + 2 + \sum_{i=1}^p d_i^2\right)$. By definition of pathos entire total block graph $P_{etb}(T)$ of a tree equal to the sum of lines in $P_{T_B}(T)$ and the number of lines which lie on block points b_i of $T_B(T)$ from the pathos points P_i , which are equal to q . Thus the number

of lines in $P_{etb}(T) = \left(q + 2 + \sum_{i=1}^p d_i^2 \right) = \left(2q + 2 + \sum_{i=1}^p d_i^2 \right)$. \square

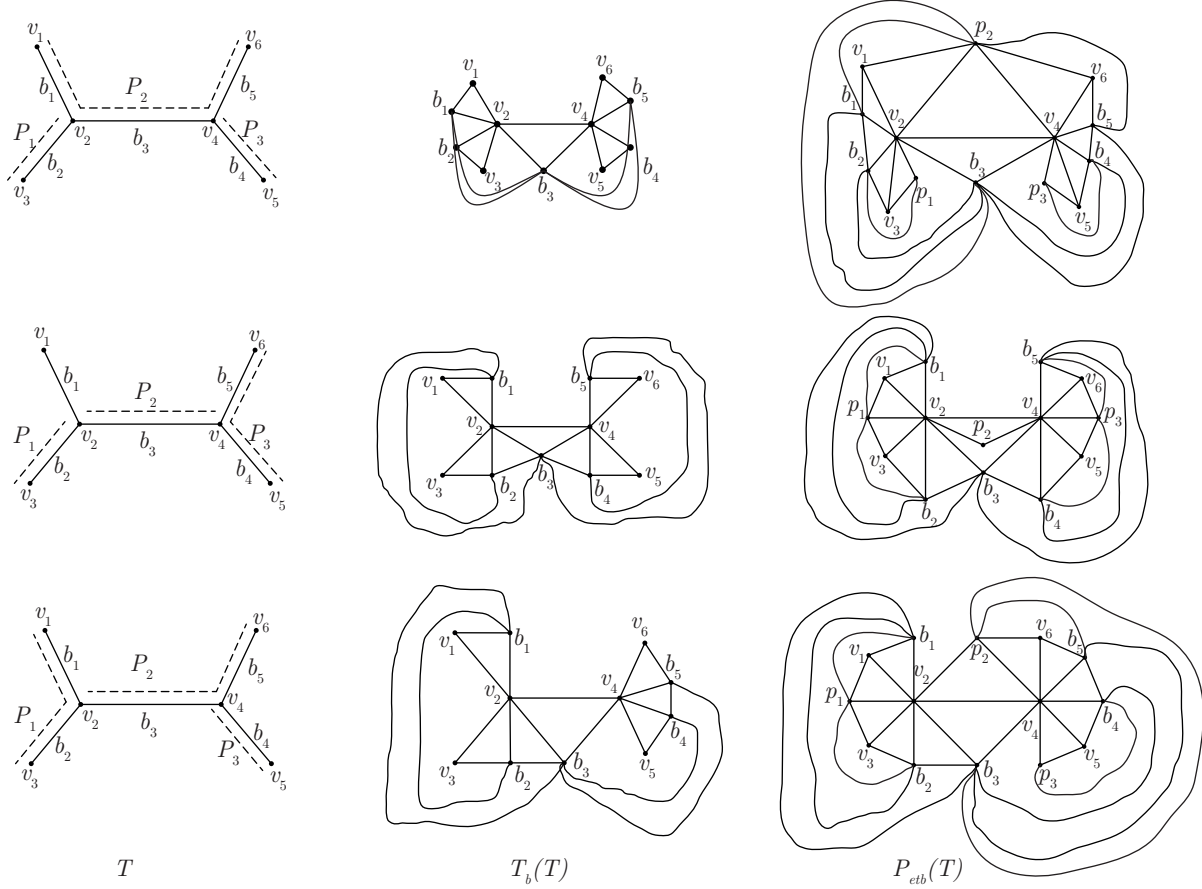


Figure 2

§5. Planar Pathos Entire Total Block Graphs

A criterion for pathos entire total block graph to be planar is presented in our next theorem.

Theorem 5.1 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_{etb}(T)$ is planar. Then by Theorem D, each cut point of T lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cut points instead of number of blocks incident with the cut points. Now suppose if $\Delta(T) \leq 3$, then by Theorem D, $T_b(T)$ is planar. Let $\{b_1, b_2, b_3, \dots, b_{p-1}\}$ be the blocks of T with p points such that $b_1 = e_1, b_2 = e_2, \dots, b_{p-1} = e_{p-1}$ and P_i be the number of pathos of T . Now $V[P_{etb}(T)] = V(G) \cup b_1, b_2, b_3, \dots, b_{p-1} \cup \{P_i\}$. By Theorem D, and by the definition of pathos,

the embedding of $P_{etb}(T)$ in any plane gives a planar $P_{etb}(T)$.

Conversely, Suppose $\Delta(T) \geq 4$ and assume that $P_{etb}(T)$ is planar. Then there exists at least one point of degree 4, assume that there exists a vertex v such that $\deg v = 4$. Then in $T_B(T)$, this point together with the block points form k_5 as an induced subgraph. Further the corresponding pathos point which is adjacent to the $V(T)$ in $T_B(T)$ which gives $P_{etb}(T)$ in which again k_5 as an induced subgraph, a contradiction to the planarity of $P_{etb}(T)$. This completes the proof. \square

The following theorem results the minimally nonouterplanar $P_{etb}(T)$.

Theorem 5.2 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is minimally nonouterplanar if and only if $T = k_2$.*

Proof Suppose $T = k_3$ and $P_{etb}(T)$ is minimally nonouterplanar. Then $T_B(T) = k_4$ and one can easily verify that, $i(P_{etb}(T)) > 1$, a contradiction. Further we assume that $T = K_{1,2}$ and $P_{etb}(T)$ is minimally outerplanar, then $T_B(T)$ is $W_p - x$, where x is outer line of W_p . Since $K_{1,2}$ has exactly one pathos, this point together with $W_p - x$ gives W_{p+1} . Also in $P_{etb}(T)$ and by definition of $P_{etb}(T)$ there are two more lines joining the pathos points there by giving W_{p+3} . Clearly, $P_{etb}(T)$ is nonminimally nonouterplanar, a contradiction.

For the converse, if $T = k_2$, $T_B(T) = k_3$ and $P_{etb}(T) = K_4$ which is a minimally nonouterplanar. This completes the proof of the theorem. \square

Now we have a pathos entire total block graph of a path $p \geq 2$ point as shown in the below remark.

Remark 5.1 For any non-trivial path with $p \geq 2$ points, $i[P_{etb}(T)] = 2p - 3$. The next theorem gives a nonminimally nonouterplanar $P_{etb}(T)$.

Theorem 5.3 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is nonminimally nonouterplanar except for $T = k_2$.*

Proof Assume T is not a path. We consider the following cases.

Case 1 Suppose T is a tree with $\Delta(T) \geq 3$. Then there exists at least one point of degree at least 3. Assume v be a point of degree 3. Clearly, $T = K_{1,3}$. Then by the Theorem F, $i[T_B(T)] > 1$. Since $T_B(T)$ is a subgraph of $P_{etb}(T)$. Clearly, $i(P_{etb}(T)) \geq 2$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar.

Case 2 Suppose T is a path with p points and for $p > 2$ points. Then by Remark 5.1, $i[P_{etb}(T)] > 1$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar. \square

In the following theorem we characterize the noneulerian $P_{etb}(T)$.

Theorem 5.4 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is noneulerian.*

Proof We consider the following cases.

Case 1 Suppose T is a path P_n with n points. Now for $n = 2$ and 3 points as follows. For $p = 2$ points, then $P_{etb}(T) = K_4$, which is noneulerian. For $p = 3$ points, then $P_{etb}(T)$ is a wheel W_6 together with two lines joining the non adjacent points in which one point is common for these two lines as shown in the Figure 3, which is noneulerian.

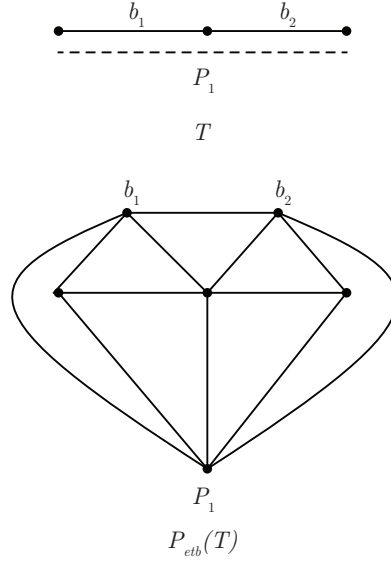


Figure 3

For $p \geq 4$ points, we have a path $P : v_1, v_2, v_3, \dots, v_p$. Now in path each line is a block. Then $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \dots, v_{p-1}v_p = e_{p-1} = b_{p-1}$, $\forall e_{p-1} \in E(G)$, and $\forall b_{p-1} \in V[T_B(P)]$. In $T_B(P)$, the degree of each point is even except b_1 and b_{p-1} . Since the path P has exactly one pathos which is a point of $P_{etb}(P)$ and is adjacent to the points $v_1, v_2, v_3, \dots, v_p$, of $T_B(P)$ which are of even degree, gives as an odd degree points in $P_{etb}(P)$ including odd degree points b_1 and b_{p-1} . Clearly $P_{etb}(P)$ is noneulerian.

Case 2 Suppose T is not a path. We consider the following subcases.

Subcase 2.1 Assume T has a unique point degree ≥ 3 and $T = K_{1,n}$, with n is odd. Then in $T_B(T)$ each block is a triangle such that there are n number of triangles with a common cut points k which has a degree $2n$. Since the degree of each point in $T_B(K_{1,n})$ is odd other than the cut point k which are of degrees either 2 or $n + 1$. Then $P_{etb}(T)$ eulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 4 the point k are joined by the corresponding pathos point which gives $(2n + 2)$ points of odd degree in $P_{etb}(K_{1,n})$. $P_{etb}(T)$ has n points of odd degree. Hence $P_{etb}(T)$ noneulerian.

Assume that $T = K_{1,n}$, where n is even, Then in $T_B(T)$ each block is a triangle, which are $2n$ in number with a common cut point k . Since the degree of each point other than k is either 2 or $(n + 1)$ and the degree of the point k is $2n$. One can easily verify that $T_B(K_{1,n})$ is noneulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 5 the point k are joined by the corresponding pathos point which gives $(n + 2)$ points of odd degree in $P_{etb}(T)$.

Hence $P_{etb}(T)$ noneulerian.

Subcase 2.2 Assume T has at east two points of degree ≥ 3 . Then $V[T_B(T)] = V(G) \cup b_1, b_2, b_3, \dots, b_p, \forall b_p \in E(G)$. In $T_B(T)$, each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in $P_{etb}(T)$ gives degree 3, From case 1, Tree T has at least 4 points and by Corollary [A], $P_{etb}(T)$ has at least two points of degree 3. Hence $P_{etb}(T)$ is noneulerian. \square

In the next theorem we characterize the hamiltonian $P_{etb}(T)$.

Theorem 5.5 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is hamiltonian.*

Proof we consider the following cases.

Case 1 Suppose T is a path with $\{u_1, u_2, u_3, \dots, u_n\} \in V(T)$ and $b_1, b_2, b_3, \dots, b_m$ be the number of blocks of T such that $m = n - 1$. Then it has exactly one path of pathos. Now point set of $T_B(T)$, $V[T_B(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\}$. Since given graph is a path then in $T_B(T)$, $b_1 = e_1, b_2 = e_2, \dots, b_m = e_m$, such that $b_1, b_2, b_3, \dots, b_m \subset V[T_B(T)]$. Then by the definition of total block graph, $\{u_1, u_2, \dots, u_{m-1}, u_m\} \cup \{b_1, b_2, \dots, b_{m-1}, b_m\} \cup \{b_1 u_1, b_2 u_2, \dots, b_m u_{n-1}, b_m u_n\}$ form line set of $T_B(T)$ (see Figure 4).

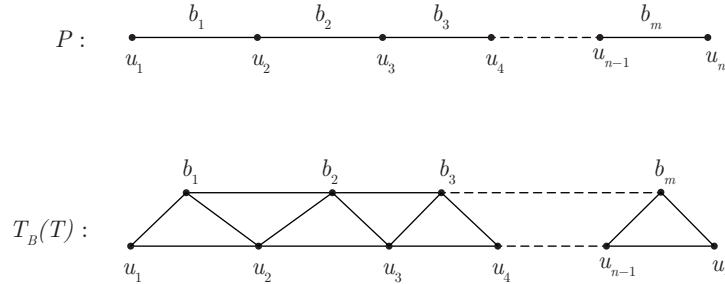


Figure 4

Now this path has exactly one pathos say w . In forming pathos entire total block graph of a path, the pathos w becomes a point, then $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\} \cup \{w\}$ and w is adjacent to all the points $\{u_1, u_2, \dots, u_n\}$ shown in the Figure 5.

In $P_{etb}(T)$, the hamiltonian cycle $w, u_1, b_1, u_2, b_2, u_3, b_3, \dots, u_{n-1}, b_m, u_n, w$ exist. Clearly the pathos entire total block graph of a path is hamiltonian graph.

Case 2 Suppose T is not a path. Then T has at least one point with degree at least 3. Assume that T has exactly one point u such that degree > 2 . Now we consider the following subcases of Case 2.

Subcase 2.1 Assume $T = K_{1,n}, n > 2$ and is odd. Then the number of paths of pathos are $\frac{n+1}{2}$. Let $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{m-1}\}$. By the definition of pathos total block graph. By the definition $P_{etb}(T)$ $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n+1/2}\}$. Then there exists a cycle containing the points of By the definition of $P_{etb}(T)$ as

$p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \dots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

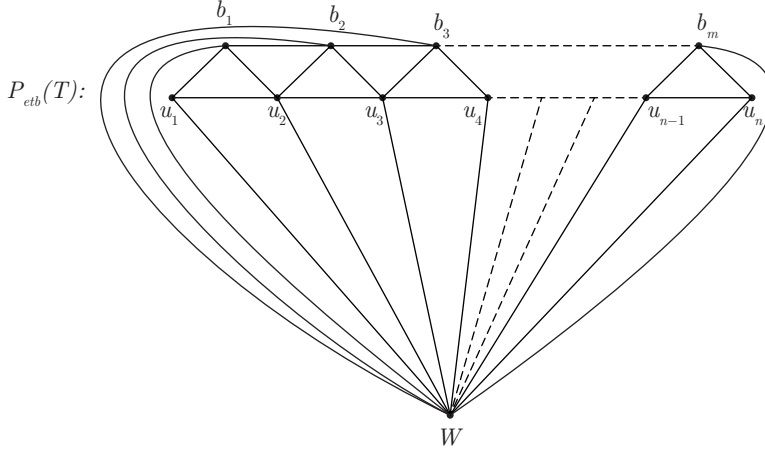


Figure 5

Subcase 2.2 Assume $T = K_{1,n}, n > 2$ and is even. Then the number of path of pathos are $\frac{n}{2}$, then $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\}$. By the definition of pathos entire total block graph $P_{etb}(T)$ of a tree T . $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n/2}\}$. Then there exist a cycle containing the points of $P_{etb}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \dots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

Suppose T is neither a path nor a star, then T contains at least two points of degree > 2 . Let $u_1, u_2, u_3, \dots, u_n$ be the points of degree ≥ 2 and $v_1, v_2, v_3, \dots, v_m$ be the end points of T . Since end block is a line in T , and denoted as b_1, b_2, \dots, b_k , then $V[T_B(T)] = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_m\} \cup \{b_1, b_2, \dots, b_k\}$, and tree T has p_i pathos points, $i > 1$ and each pathos point is adjacent to the point of T where the corresponding pathos lie on the points of T . Let $\{p_1, p_2, \dots, p_i\}$ be the pathos points of T . Then there exists a cycle C containing all the points of $P_{etb}(T)$, as $p_1, v_1, b_1, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, v_{m-1}, b_{n-1}, b_n, v_m, \dots, p_1$. Hence $P_{etb}(T)$ is a hamiltonian cycle. Clearly, $P_{etb}(T)$ is a hamiltonian graph. \square

In the next theorem we characterize $P_{etb}(T)$ in terms of crossing number one.

Theorem 5.6 For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T has crossing number one if and only if $\Delta(T) \leq 4$, and there exist a unique point in T of degree 4.

Proof Suppose $P_{etb}(T)$ has crossing number one. Then it is nonplanar. Then by Theorem 5.1, we have $\Delta(T) \leq 4$. We now consider the following cases.

Case 1 Assume $\Delta(T) = 5$. Then by Theorem [F], $T_B(T)$ is nonplanar with crossing number more than one. Since $T_B(T)$ is a subgraph of $P_{etb}(T)$. Clearly $cr(P_{etb}(T)) > 1$, a contradiction.

Case 2 Assume $\Delta(T) = 4$. Suppose T has two points of degree 4. Then by Theorem F, $T_B(T)$ has crossing number at least two. But $T_B(T)$ is a subgraph of $P_{etb}(T)$. Hence $cr(P_{etb}(T)) > 1$,

a contradiction.

Conversely, suppose T satisfies the given condition and assume T has a unique point v of degree 4. The lines which are blocks in T such that they are the points in $T_B(T)$. In $T_B(T)$, these block points and a point v together forms an induced subgraph as K_5 . In forming $P_{etb}(T)$, the pathos points are adjacent to at least two points of this induced subgraph. Hence in all these cases the $cr(P_{etb}(T)) = 1$. This completes the proof. \square

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On Folding of Groups

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Abstract: The aim of our study is to give a definition of the folding of groups and study the folding of some types of groups such as cyclic groups and dihedral groups, also we discussed the folding of direct product of groups. Finally the folding of semigroups are investigated.

Key Words: Folding, multi-semigroup, multi-group, group, commutative semigroup.

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§1. Introduction

In the last two decades there has been tremendous progress in the theory of folding. The notion of isometric folding is introduced by S. A. Robertson who studied the stratification determined by the folds or singularities [10]. The conditional foldings of manifolds are defined by M. El-Ghoul in [8]. Some applications on the folding of a manifold into it self was introduced by P. Di. Francesco in [9]. Also a folding in the algebra's branch introduced by M.El-Ghoul in [7]. Then the theory of isometric foldings has been pushed and also different types of foldings are introduced by E. El-Kholy and others [1,2,5,6].

Definition 1.1([4]) *A non empty set G on which is defined $s \geq 1$ associative binary operations $*$ is called a multi-semigroup, if for all $a, b \in G, a * b \in G$, particularly, if $s = 1$, such a multi-semigroup is called a semigroup.*

Example 1 $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$ is a semigroup under multiplication $p, p \in \mathbb{Z}^+$.

Definition 1.2([4]) *A subset H is a subsemigroup of G if H is closed under the operation of G ; that is if $a * b \in H$ for all $a, b \in H$.*

Definition 1.3 *A multi-group (G, \mathcal{O}) is a non empty set G together with a binary operation set \mathcal{O} on G such that for $*$ in \mathcal{O} , the following conditions hold:*

- (1) $\forall a, b \in G$ then $a * b \in G$.
- (2) There exists an element $e \in G$ such that $a * e = e * a = a$, for all $a \in G$.
- (3) For $a \in G$ there is an element a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$.

Particularly, if $|\mathcal{O}| = 1$, such a multi-group is called a group and denoted by G

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A group G is called *Abelian* if $a * b = b * a$ for all $a, b \in G$. The order of a group is its cardinality, i.e., the number of its elements. We denote the order of a group G by $|G|$.

Definition 1.4([3]) *The group G is called a cyclic group of order n , if there exists an element $g \in G$, such that $G = \langle g \rangle = \{g \mid g^n = 1\}$. In this case g is called a generator of G .*

Definition 1.5([3]) *The dihedral group D_{2n} of order $2n$, is defined in the following equivalent ways: $D_{2n} = \{a, b \mid a^2 = b^n = 1, bab = a\}$.*

Definition 1.6([3]) *A subset H is a subgroup of G , $H \leq G$, if H is closed under the operation of G ; that is if $a * b \in H$ for all $a, b \in H$.*

Definition 1.7([3]) *The trivial subgroup of any group is the subgroup $\{e\}$, consisting of just the identity element.*

Definition 1.8([3]) *A subgroup of a group G that does not include the entire group itself is known as a proper subgroup, denoted by $H < G$.*

Theorem 1.1 *Every subgroup of a cyclic group is cyclic.*

§2. Group Folding

In this section we give the notions of group folding and discuss the folding of some kinds of groups

Definition 2.1 *Let G_1, G_2 are two groups, The group folding $\mathbf{g.f}$ of G_1 into G_2 is the map $f : (G_1, *) \rightarrow (G_2, \circ)$ st.*

$$\forall a \in G_1, f(a) = b, f(a^{-1}) = b^{-1}$$

where $b \in G_2$ and $f(G_1)$ is subgroup of G_2 .

Definition 2.2 *The set of singularities $\sum f$ is the set of elements $a_i \in G$ such that $f(a_i) = a_i$.*

Definition 2.3 *A group folding is called good group folding $\mathbf{g.g.f}$ if H is non trivial subgroup of G .*

Proposition 1.1 *The limit of group folding of any group is $\{e\}$.*

Proof Let G be any group and since any group has two trivial subgroups $(G, *)$, $(\{e\}, *)$, where $(\{e\}, *)$ is the minimum subgroup of G . Then if we define a sequence of group folding $f_i : (G, *) \rightarrow (G, *)$, such that $f(a_i) = b_i$, $f(a_i^{-1}) = b_i^{-1}$. We found that $\lim_{i \rightarrow \infty} f_i(G) = \{e\}$. \square

Theorem 2.1 *Let G be cyclic group $G = \{g : g^p = 1\}$, where p is prime. Then there is no $\mathbf{g.g.f}$ map can be defined on G .*

Proof Given G is cyclic group of order p , since p is prime. Then there is no proper subgroup can be found in G . Hence we can not able to defined any $\mathbf{g.g.f}$ map on group G . \square

Theorem 2.2 Let G be cyclic group $G = \{g : g^p = 1\}$, where $p = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$ and q_1, q_2, \dots, q_n are distinct prime number. Then the limit of $\mathbf{g.g.f}$ map of G is the proper subgroup, $H = \left\{g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1\right\}$.

Proof Since G be cyclic group and $|G| = p = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$, q_1, q_2, \dots, q_n are distinct prime number. Then there exist proper subgroup $H < G$ such that $|H| = k$. So we can defined $\mathbf{g.g.f}$ map $f_1 : G \longrightarrow G$, such that $f_1(G) = H$, after this there exist two cases.

Case 1. If k is prime number then we can not found a proper subgroup of H and so $\lim_{i \rightarrow \infty} f_i(G) = H$ and $k = q_i, 1 \leq i \leq n, f_i(G) = H = \left\{g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1\right\}$.

Case 2. If k is not prime. Then $k = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}, m < n$, hence there exist a proper subgroup $\bar{H} < H, |\bar{H}| = \bar{k}$. Thus we can defined a $\mathbf{g.g.f}$ map $f_2 : H \longrightarrow H$, such that $f_2(H) = \bar{H}$. Again if $|\bar{H}|$ is prime. Then $\lim_{i \rightarrow \infty} f_i(G) = \bar{H}, \bar{k} = q_i, 1 \leq i \leq m, f_i(G) = \bar{H} = \left\{g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1\right\}$ or, if $|\bar{H}| = \bar{k}$ is not prime. Then we can repeat the Case II again. Finally the limit of $\mathbf{g.g.f}$ of G is the subgroup $H = \left\{g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1\right\}$. \square

Corollary 2.1 If $p = q^\alpha$ then the limit of $\mathbf{g.g.f}$ map of G is a subgroup $H = \left\{g^q : (g^q)^{\frac{p}{q}} = 1\right\}$.

Example 2 Let $G = \{g : g^{12} = 1\}$ be a cyclic group, $|G| = 12 = 2^2 \cdot 3$, so we can defined a $\mathbf{g.g.f}$ map as the following $f_1 : G \longrightarrow G$, such that:

$$\begin{aligned} f_1(1) &= 1, f_1(g^2) = g^2, f_1(g^{10}) = g^{10}, \\ f_1(g^4) &= g^4, f_1(g^6) = g^6, f_1(g^8) = g^8, \\ f_1(g^5) &= g^2, f_1(g^7) = g^{10}, \\ f_1(g^3) &= g^4, f_1(g^9) = g^8, \\ f_1(g) &= g^6, f_1(g^{11}) = g^6 \end{aligned}$$

and $f_1(G) = H = \{1, g^2, g^4, g^6, g^8, g^{10}\}$, since the order of H is not prime then we can defined a $\mathbf{g.g.f}$ map as the following $f_2 : H \longrightarrow H$ such that

$$\begin{aligned} f_2(1) &= 1, f_2(g^6) = 1 \\ f_2(g^2) &= g^4, f_2(g^{10}) = g^8, \\ f_2(g^4) &= g^4, f_2(g^8) = g^8, \end{aligned}$$

and $f_2(H) = \bar{H} = \{1, g^4, g^8\}$ is proper subgroup of H . Since the order of \bar{H} is prime, then $\lim_{i \rightarrow \infty} f_i(G) = \bar{H} = \{g^4 | (g^4)^3 = 1\}$.

Proposition 2.2 For any dihedral group $D_{2n} = \{a, b \mid a = b^n = 1, bab = 1\}$, we can defined a $\mathbf{g.g.f}$ map.

Theorem 2.3 Let D_{2n} be a dihedral group, where $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ and q_1, q_2, \dots, q_m are distinct prime number. Then the limit of $\mathbf{g.g.f}$ of D_{2n} is one of these $H_i = \{1, ab^i\}, i = 1, 2, \dots, n$ or $H_i = \left\{b^{q_i} | (b^{q_i})^{\frac{n}{q_i}} = 1\right\}, i = 1, 2, \dots, m$.

Proof Let $D_{2n} = \{a, b \mid a = b^n = 1, bab = 1\}$ be a dihedral group of order $2n$. Then the group D_{2n} has n proper subgroups $H_i = \{1, ab^i\}$, $i = 1, 2, \dots, n$ of order 2, and so we can defined a **g.g.f** map $f : D_{2n} \longrightarrow D_{2n}$, such that $f(D_{2n}) = H_i$.

Also since the dihedral group has proper cyclic subgroup $H = \{b \mid b^n = 1\}$. Then we can defined a **g.g.f** map $f : D_{2n} \longrightarrow D_{2n}$, such that $f(D_{2n}) = \{b \mid b^n = 1\}$, and since the subgroup $H = \{b \mid b^n = 1\}$ be cyclic group of order $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_m^{\alpha_m}$. Thus by applying the Theorem 2.2 we get that $\lim_{i \rightarrow \infty} f_i(D_{2n}) = H = \left\{ b^{q_i} \mid (b^{q_i})^{\frac{n}{q_i}} = 1 \right\}, i = 1, 2, \dots, m$. \square

§3. Group Folding of the Direct Product of Groups

In this section we discuss the group folding of direct product of groups. Let $f_1 : (G_1, *) \rightarrow (G_1, *)$, $f_2 : (G_2, *) \rightarrow (G_2, *)$ are two **g.g.f** maps. Then we define the direct product of the f_1, f_2 as the following:

$$\begin{aligned} f_1 \times f_2 : (G_1 \times G_2, *) &\rightarrow (G_1 \times G_2, *) \\ \forall a \in G_1, b \in G_2, (f_1 \times f_2)(a, b) &= (f_1(a), f_2(b)) \end{aligned}$$

Theorem 3.1 *The direct product of two **g.g.f** maps are **g.g.f** map also, but the converse is not always true.*

Proof Since $f_1 : (G_1, *) \rightarrow (G_1, *)$, $f_2 : (G_2, *) \rightarrow (G_2, *)$ are two **g.g.f** maps then there exist two proper subgroups H_1, H_2 of G_1, G_2 respectively, such that $f_1(G_1) = H_1$, $f_2(G_2) = H_2$. As the direct product of the any two proper subgroups are proper subgroups. Hence $H_1 \times H_2 < G_1 \times G_2$. Now we will proof that the map, $f^* = f_1 \times f_2 : G_1 \times G_2 \rightarrow G_1 \times G_2$ is **g.g.f** map. Let $a, b \in G_1, G_2$ and $c, d \in H_1, H_2$ respectively, then we have:

$$\begin{aligned} f^*(a, b) &= (f_1 \times f_2)(a, b) = (f_1(a), f_2(b)) = (c, d) \in H_1 \times H_2, \\ f^*(a^{-1}, b^{-1}) &= (f_1 \times f_2)(a^{-1}, b^{-1}) = (c^{-1}, d^{-1}) \in H_1 \times H_2. \end{aligned}$$

Hence the map f^* is **g.g.f** map. To proof the converse is not true, let $G_1 = \{g_1 \mid g_1^2 = 1\}$, $G_2 = \{g_2 \mid g_2^3 = 1\}$ be cyclic groups and since the order of them are prime then from Theorem 2.1, we can not able to define a **g.g.f** map of them. But the direct product of G_1, G_2 is

$$G_1 \times G_2 = \{(1, 1), (1, g_2), (1, g_2^2), (g_1, 1), (g_1, g_2), (g_1, g_2^2)\}.$$

and since the $|G_1 \times G_2| = 6$. Hence $G_1 \times G_2$ has a proper subgroup $H = \{(1, 1), (1, g_2)\}$ and so we can define the **g.g.f** map $f^* : G_1 \times G_2 \rightarrow G_1 \times G_2$ as the following:

$$\begin{aligned} f^*(1, 1) &= (1, 1), \\ f^*(1, g_2) &= (1, g_2), \quad f^*(g_1, 1) = (1, 1), \\ f^*(g_1, g_2) &= (1, g_2), \\ f^*(g_1, g_2^2) &= (1, g_2). \end{aligned}$$

This completes the proof. \square

§4. Folding of Semigroups

In this section we will be discuss the folding of semigroups into itself. Let $(G, *)$ be a commutative semigroup with identity 1, i.e. a monoid.

Definition 4.1([4]) *A nonzero element a of a semigroup G is a left zero divisor if there exists a nonzero b such that $a * b = 0$. Similarly, a is a right zero divisor if there exists a nonzero element $c \in G$ such that $c * a = 0$.*

The element a is said to be a *zero divisor* if it is both a left and right zero divisor. we will denote to the set of all zero-divisors by $Z(G)$, and the set of all elements which have the inverse by $I(G)$.

Definition 4.2 *The zero divisor folding of the semigroup G , **z.d.f**, is the map $f_z : (G, *) \rightarrow (G, *)$, st.*

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x * a = 0, x, a \neq 0 \\ x & \text{if } x * a \neq 0, x, a \neq 0 \end{cases}$$

where $a \in Z(G)$.

Note that $f_z(G)$ may be semigroup or not. We will investigate the zero divisor folding for \mathbb{Z}_p semigroups.

Definition 4.3 *Let \mathbb{Z}_p be a semigroup under multiplication modulo p . The **z.d.f** map of \mathbb{Z}_p is the map $f_z : (G, \cdot) \rightarrow (G, \cdot)$, st.*

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0 \\ q & \text{if } xq = 0, x, q \neq 0 \\ x & \text{if } xq \neq 0, x, q \neq 0 \end{cases}$$

where $q \in Z(\mathbb{Z}_p)$, is the greatest divisor of p .

Proposition 4.1 *If the order of \mathbb{Z}_p is prime. Then $f_z(G) = G$, i.e. f_z is identity map.*

Proof Since the order of semigroup \mathbb{Z}_p is prime. Then the semigroup \mathbb{Z}_p has not got any zero divisor, and so the **z.d.f** which can defined on \mathbb{Z}_p the identity map $f_z(x) = x$, for all $x \in \mathbb{Z}_p$. \square

Theorem 4.1 *Let \mathbb{Z}_p be semigroup of order p , then **z.d.f** of \mathbb{Z}_p into itself is a subsemigroup under multiplication modulo p . Has one zero divisor if $4 \mid p$ or has not any zero divisor if $4 \nmid p$.*

Proof Let \mathbb{Z}_p be semigroup under multiplication modulo p . Then \mathbb{Z}_p consists of two subsets $Z(\mathbb{Z}_p)$, $I(\mathbb{Z}_p)$.

Case 1. If p is even, then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{p}{2} & \text{if } x \in Z(\mathbb{Z}_p), x \text{ is even}; \\ x & \text{if } x \in Z(\mathbb{Z}_p), x \text{ is odd or } x \in I(\mathbb{Z}_p), \end{cases}$$

where $\frac{p}{2}$ is the greatest divisor of p . Hence $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, \frac{p}{2}, x_1, \dots, x_n\}$, where x_1, \dots, x_n are odd zero divisors and $x_i x_j \neq 0$, for all $i, j = 1, \dots, n$. Notice that $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, \frac{p}{2}, x_1, \dots, x_n\}$ is subsemigroup under multiplication modulo p . If $4 \mid p$ then $\frac{p}{2} \cdot \frac{p}{2} = 0$, hence the subsemigroup $f_z(\mathbb{Z}_p)$ has one zero divisor $\frac{p}{2}$. Otherwise, if $4 \nmid p$ then $\frac{p}{2}, \frac{p}{2} \neq 0$. And so $H = I(\mathbb{Z}_p) \cup \{0, \frac{p}{2}, x_1, \dots, x_n\}$ is subsemigroup under multiplication modulo p without any zero divisor.

Case 2. If p is odd, then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ q & \text{if } x \in Z(\mathbb{Z}_p), x \text{ is odd}; \\ x & \text{if } x \in Z(\mathbb{Z}_p), x \text{ is even or } x \in I(\mathbb{Z}_p), \end{cases}$$

where q is the greatest divisor of p . Hence $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, q, x_1, \dots, x_n\}$, where x_1, \dots, x_n are even zero divisors and $x_i x_j \neq 0$ for all $i, j = 1, \dots, n$. Notice that $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, q, x_1, \dots, x_n\}$ is subsemigroup under multiplication modulo p , and since p is odd. Then $f_z(\mathbb{Z} - p)$ has not any zero divisor. \square

Corollary 4.1 *If \mathbb{Z}_p be semigroup under multiplication modulo p and $p = (q)^m, m \in \mathbb{N}$. Then $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, q^{m-1}\}$ is subsemigroup under multiplication modulo p .*

Example 3. (1) Let $\mathbb{Z}_{10} = \{0, 1, 2, 3, \dots, 9\}$ be semigroup under multiplication modulo 10. Since $p = 10$ is even then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ 5 & \text{if } x \in \{2, 4, 8, 6\}; \\ x & \text{if } x \in \{1, 3, 7, 9, 5\}. \end{cases}$$

Thus $f_z(\mathbb{Z}_{10}) = \{0, 1, 3, 7, 9, 5\}$. Obviously, $f_z(\mathbb{Z}_{10})$ is a subsemigroup under multiplication modulo 10. Since $4 \nmid 10$, then $f_z(\mathbb{Z}_{10})$ has not any zero divisor.

(2) Let $\mathbb{Z}_{12} = \{0, 1, 2, 3, \dots, 11\}$ be a semigroup under multiplication modulo 12. Since $p = 12$ is even then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ 6 & \text{if } x \in \{2, 4, 6, 8, 10\}; \\ x & \text{if } x \in \{1, 3, 5, 7, 9\}. \end{cases}$$

Hence $f_z(\mathbb{Z}_{12}) = \{0, 1, 3, 7, 9, 5, 6\}$. Obviously, $f_z(\mathbb{Z}_{12})$ is a subsemigroup under multiplication modulo 12 and 6 is a zero divisor.

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On Set-Semigraceful Graphs

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Abstract: This paper studies certain properties of set-semigraceful graphs and obtain certain bounds for the order and size of such graphs. More set-semigraceful graphs from given ones are also obtained through various graph theoretic methods.

Key Words: Set-indexer, set-graceful, set-semigraceful.

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§1. Introduction

In 1986, B. D. Acharya introduced the concept of set-indexer of a graph G which is an assignment of distinct subsets of a finite set X to the vertices of G subject to certain conditions. Based on this, the notions of set-graceful and set-semigraceful graphs were derived. Later many authors have studied about set-graceful graphs and obtained many significant results.

This paper sheds more light on set-semigraceful graphs. Apart from many classes of set-semigraceful graphs, several properties of them are also investigated. Certain bounds for the order and size of set-semigraceful graphs are derived. More set-semigraceful graphs from given ones are also obtained through various techniques of graph theory.

§2. Preliminaries

In this section we include certain definitions and known results needed for the subsequent development of the study. Throughout this paper, l , m and n stand for natural numbers without restrictions unless and otherwise mentioned. For a nonempty set X , the set of all subsets of X is denoted by 2^X . We always denote a graph under consideration by G and its vertex and edge sets by V and E respectively and G' being a subgraph of a graph G is denoted by $G' \subseteq G$. When G' is a proper subgraph of G we denote it by $G' \subset G$. By the term graph we mean a simple graph and the basic notations and definitions of graph theory are assumed to be familiar to the readers.

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Definition 2.1([1]) *Let $G = (V, E)$ be a given graph and X be a nonempty set. Then a mapping $f : V \rightarrow 2^X$, or $f : E \rightarrow 2^X$, or $f : V \cup E \rightarrow 2^X$ is called a set-assignment or set-valuation of the vertices or edges or both.*

Definition 2.2([1]) *Let G be a given graph and X be a nonempty set. Then a set-valuation $f : V \cup E \rightarrow 2^X$ is a set-indexer of G if*

1. $f(uv) = f(u) \oplus f(v), \forall uv \in E$, where ' \oplus ' denotes the binary operation of taking the symmetric difference of the sets in 2^X
2. the restriction maps $f|_V$ and $f|_E$ are both injective.

In this case, X is called an indexing set of G . Clearly a graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the set-indexing number of G , denoted by $\gamma(G)$. The set-indexing number of trivial graph K_1 is defined to be zero.

Theorem 2.3([1]) *Every graph has a set-indexer.*

Theorem 2.4([1]) *If X is an indexing set of $G = (V, E)$. Then*

- (i) $|E| \leq 2^{|X|} - 1$ and
- (ii) $\lceil \log_2(|E| + 1) \rceil \leq \gamma(G) \leq |V| - 1$, where $\lceil \cdot \rceil$ is the ceiling function.

Theorem 2.5([1]) *If G' is subgraph of G , then $\gamma(G') \leq \gamma(G)$.*

Theorem 2.6([3]) *The set-indexing number of the Heawood Graph is 5.*

Theorem 2.7([2]) *The set-indexing number of the Peterson graph is 5.*

Theorem 2.8([13]) *If G is a graph of order n , then $\gamma(G) \geq \gamma(K_{1,n-1})$.*

Theorem 2.9([13]) $\gamma(K_{1,m}) = n + 1$ if and only if $2^n \leq m \leq 2^{n+1} - 1$.

Theorem 2.10([15]) $\gamma(P_m) = n + 1$, where $2^n \leq m \leq 2^{n+1} - 1$.

Definition 2.11([17]) *The double star graph $ST(m, n)$ is the graph formed by two stars $K_{1,m}$ and $K_{1,n}$ by joining their centers by an edge.*

Theorem 2.12([16]) *For a double star graph $ST(m, n)$ with $|V| = 2^l, l \geq 2$,*

$$\gamma(ST(m, n)) = \begin{cases} l & \text{if } m \text{ is even,} \\ l + 1 & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 2.13([16]) $\gamma(ST(m, n)) = l + 1$ if $2^l + 1 \leq |V| \leq 2^{l+1} - 1; l \geq 2$.

Theorem 2.14([1]) $\gamma(C_5) = 4$.

Theorem 2.15([1]) $\gamma(C_6) = 4$.

Definition 2.16([11]) *The join $K_1 \vee P_{n-1}$ of K_1 and P_{n-1} is called a fan graph and is denoted by F_n .*

Theorem 2.17([15]) $\gamma(F_n) = m + 1$, where $m = \lceil \log_2 n \rceil$ and $n \geq 4$.

Definition 2.18([6]) *The double fan graph is obtained by joining P_n and K_2 .*

Theorem 2.19([1]) $\gamma(K_n) = \begin{cases} n-1 & \text{if } 1 \leq n \leq 5 \\ n-2 & \text{if } 6 \leq n \leq 7 \end{cases}$

Theorem 2.20([1]) $\gamma(K_n) = \begin{cases} 6 & \text{if } 8 \leq n \leq 9 \\ 7 & \text{if } 10 \leq n \leq 12 \\ 8 & \text{if } 13 \leq n \leq 15 \end{cases}$

Definition 2.21([10]) *For a graph G , the splitting graph $S'(G)$ is obtained from G by adding for each vertex v of G , a new vertex say v' so that v' is adjacent to every vertex that is adjacent to v .*

Definition 2.22([4]) *An n -sun is a graph that consists of a cycle C_n and an edge terminating in a vertex of degree one attached to each vertex of C_n .*

Definition 2.23([8]) *The wheel graph with n spokes, W_n , is the graph that consists of an n -cycle and one additional vertex, say u , that is adjacent to all the vertices of the cycle.*

Definition 2.24([11]) *The helm graph H_n is the graph obtained from a wheel $W_n = C_n \vee K_1$ by attaching a pendant edge at each vertex of C_n .*

Definition 2.25([10]) *The twing is a graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.*

Definition 2.26([3]) *The triangular book is the graph $K_2 \vee N_m$, where N_m is the null graph of order m .*

Definition 2.27([6]) *The Gear graph is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle.*

Definition 2.28([10]) *Embedding is a mapping ζ of the vertices of G into the set of vertices of a graph H such that the subgraph induced by the set $\{\zeta(u) : u \in V(G)\}$ is isomorphic to G ; for all practical purposes, we shall assume then that G is indeed a subgraph of H .*

Definition 2.29([1]) *A graph G is said to be set-graceful if $\gamma(G) = \log_2(|E| + 1)$ and the corresponding optimal set-indexer is called a set-graceful labeling of G .*

Theorem 2.30([9]) *Every cycle C_{2^n-1} , $n \geq 2$ is set-graceful.*

Theorem 2.31([10]) *$K_{3,5}$ is not set-graceful.*

Theorem 2.32([15]) $\gamma(H_{2^n-1}) = n + 2$ for $n \geq 2$.

Theorem 2.33([1]) *The star $K_{1,2^n-1}$ is set-graceful.*

§3. Certain Properties of Set-Semigraceful Graphs

In this section we derive certain properties of set-semigraceful graphs and obtain certain bounds for the order and size of such graphs.

Definition 3.1([1]) *A graph G is said to be set-semigraceful if $\gamma(G) = \lceil \log_2(|E| + 1) \rceil$.*

Remark 3.2 (1) The Heawood graph is set-semigraceful by Theorem 2.6.

(2) The stars are set-semigraceful by Theorem 2.9.

(3) The Paths P_n ; $n \neq 2^m$, $m \geq 2$ are set-semigraceful by Theorem 2.10.

(4) K_n is set-semigraceful if $n \in \{1, \dots, 7, 9, 12\}$ by Theorems 2.19 and 2.20.

(5) The Double Stars $ST(m, n)$; $m + n \neq 2^l$, m is odd are set-semigraceful by Theorems 2.12 and 2.13.

(6) The helm graph H_{2^n-1} is set-semigraceful by Theorem 2.32.

(7) All set graceful graphs are set-semigraceful.

(8) The path P_7 is set-semigraceful but it is not set-graceful.

(9) A graph G of size $2^m - 1$ is set-semigraceful if and only if it is set-graceful.

(10) Not all graphs of size $2^m - 1$ is set-semigraceful. For example $K_{3,5}$ is not set-semigraceful by Theorem 2.31.

The following theorem is an immediate consequence of the above definition.

Theorem 3.3 *Let G be a (p, q) -graph with $\gamma(G) = m$. Then G is set-semigraceful if and only if $2^{m-1} \leq q \leq 2^m - 1$.*

Corollary 3.4 *Let G' be a (p, q') spanning subgraph of a set-semigraceful (p, q) -graph G with $q' \geq 2^{\gamma(G)-1}$. Then G' is set-semigraceful.*

Proof The proof follows from Theorems 2.4 and 3.3. □

Remark 3.5 (1) The Peterson graph is not set-semigraceful by Theorem 2.7.

(2) The paths P_{2^m} ; $m \geq 2$ are not set-semigraceful by Theorem 2.10.

(3) The double stars $ST(m, n)$; $m + n = 2^l$, m is odd are not set-semigraceful by Theorem 2.12.

(4) K_n is not set-semigraceful if $n \in \{8, 10, 11, 13, 14, 15\}$ by Theorem 2.20.

Corollary 3.6 *Let T be a set-semigraceful tree of order p , then $2^{\gamma(T)-1} + 1 \leq p \leq 2^{\gamma(T)}$.*

While Theorem 3.3 gives bounds for the size of a set-semigraceful graph in terms of the set-indexing number, the following one gives the same in terms of its order.

Theorem 3.7 *Let G be a set-semigraceful (p, q) -graph. Then*

$$2^{\lceil \log_2 p \rceil - 1} \leq q \leq 2^{\lceil \log_2(\frac{p(p-1)}{2} + 1) \rceil} - 1.$$

Proof By Theorems 2.4, 2.5 and 2.8, we have

$$\lceil \log_2 p \rceil \leq \gamma(K_{1,p-1}) \leq \gamma(G) = \lceil \log_2(q+1) \rceil \leq \left\lceil \log_2\left(\frac{p(p-1)}{2} + 1\right) \right\rceil.$$

Now by Theorem 3.3 we have

$$2^{\lceil \log_2 p \rceil - 1} \leq q \leq 2^{\lceil \log_2(\frac{p(p-1)}{2} + 1) \rceil} - 1. \quad \square$$

Remark 3.8 The converse of Theorem 3.7 is not always true. By Theorem 2.14 we have, $\lceil \log_2(|E(C_5)| + 1) \rceil = \lceil \log_2 6 \rceil = 3 < \gamma(C_5) = 4$. But C_5 is not set-semigraceful even if $2^2 \leq |E(C_5)| \leq 2^4 - 1$ holds. Further as a consequence of the above theorem we have the graphs $C_6 \cup 3K_1$, $C_5 \cup 4K_1$ and $C_5 \cup 2K_2$ are not set-semigraceful.

Remark 3.9 By Theorem 2.5, for any subgraph G' of G , $\gamma(G') \leq \gamma(G)$. But subgraphs of a set-semigraceful graph need not be set-semigraceful. For example K_6 is set-semigraceful but the spanning subgraph C_6 of K_6 is not set-semigraceful, by Theorem 2.19.

In fact the result given by Theorem 3.3 holds for any set-semigraceful graph as we see in the following.

Theorem 3.10 *Every connected set-semigraceful $(p, p-1)$ -graph is a tree such that*

$$2^{m-1} + 1 \leq p \leq 2^m$$

and for every m , such a tree always exists.

Proof Clearly every connected $(p, p-1)$ graph T is a tree and by Theorem 3.3 we have $2^{m-1} + 1 \leq p \leq 2^m$ if T is set-semigraceful. On the other hand, for a given m , the star graph $K_{1,2^m-1}$ is set-semigraceful. \square

Theorem 3.11 *If the complete graph K_n ; $n \geq 2$ is set-semigraceful then*

$$2m - 1 \leq \gamma(K_n) \leq 2m + 1, \text{ where, } m = \lfloor \log_2 n \rfloor.$$

Proof If K_n is set-semigraceful then

$$\begin{aligned} \left\lceil \log_2 \frac{n(n-1)}{2} + 1 \right\rceil &= \gamma(K_n) \geq \left\lceil \log_2 \frac{n(n-1)}{2} \right\rceil \\ &= \lceil \log_2 n + \log_2(n-1) - \log_2 2 \rceil \\ &= \lceil \log_2 n + \log_2(n-1) - 1 \rceil. \end{aligned}$$

For any n , there exists m such that $2^m \leq n \leq 2^{m+1} - 1$ so that from above $\gamma(K_n) \geq 2m - 1$. But we have

$$\gamma(K_n) = \left\lceil \log_2 \frac{n(n-1)}{2} + 1 \right\rceil = \left\lceil \log_2 \frac{n(n-1)+2}{2} \right\rceil \leq 2m + 1.$$

Thus we have $2m - 1 \leq \gamma(K_n) \leq 2m + 1$; $m = \lfloor \log_2 n \rfloor$. \square

Remark 3.12 The converse of Theorem 3.11 is not always true. For example, by Theorem 2.20 we have

$$2 \lfloor \log_2 n \rfloor - 1 \leq \gamma(K_8) \leq 2 \lfloor \log_2 n \rfloor + 1.$$

But the complete graph K_8 is not set-semigraceful. Also by Theorems 2.20 and 3.11 we have K_{13} , K_{14} and K_{15} are not set-semigraceful.

Theorem 3.13 *If a (p, q) -graph G has a set-semigraceful labeling with respect to a set X of cardinality $m \geq 2$, there exists a partition of the vertex set $V(G)$ into two nonempty sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is at most 2^{m-1} .*

Proof Let $f : V \cup E \rightarrow 2^X$ be a set-semigraceful labeling of G with indexing set X of cardinality m . Let $V_1 = \{u \in V : |f(u)| \text{ is odd}\}$ and $V_2 = V - V_1$. We have $|A \oplus B| = |A| + |B| - 2|A \cap B|$ for any two subsets A, B of X and hence $|A \oplus B| \equiv 1 \pmod{2} \Leftrightarrow A$ and B does not belong to the same set V_i ; $i = 1, 2$. Therefore all odd cardinality subsets of X in $f(E)$ must appear on edges joining V_1 and V_2 . Consequently there exists at most 2^{m-1} edges between V_1 and V_2 . \square

Remark 3.14 In 1986, B.D.Acharya [1] conjectured that the cycle C_{2^n-1} ; $n \geq 2$ is set-graceful and in 1989, Mollard and Payan [9] settled this in the affirmative. The idea of their proof is the following:

Consider the field $GF(2^n)$ constructed by a binary primitive polynomial say $p(x)$ of degree n . Let α be a root of $p(x)$ in $GF(2^n)$. Then $GF(2^n) = \{0, 1, \alpha, \alpha^2, \dots, \alpha^{2^n-2}\}$. Now by assigning $\alpha^{i-1} \bmod p(\alpha)$, $1 \leq i \leq 2^n-1$, to the vertices v_i of the cycle $C_{2^n-1} = (v_1, \dots, v_{2^n-1}, v_1)$ we get a set-graceful labeling of C_{2^n-1} with indexing set $X = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$. Note that here $\alpha^j \bmod p(\alpha) = a_0\alpha^0 + a_1\alpha^1 + \dots + a_{n-1}\alpha^{n-1}$; $a_i = 0$ or 1 for $0 \leq i \leq n-1$ with $\alpha^0 = 1$ and we identify it as $\{a_i\alpha^i / a_i = 1; 0 \leq i \leq n-1\}$ which is a subset of X .

Theorem 3.15 *All cycles C_k with $2^n - 1 \leq k \leq 2^n + 2^{n-1} - 2$, $n \geq 3$ are set-semigraceful.*

Proof The cycle $C_{2^n-1} = (v_1, v_2, \dots, v_{2^n-1}, v_1)$ has a set-graceful labeling f as described in the above Remark 3.14. Take $l = 2^{n-1} - 1$ new vertices u_1, \dots, u_l and form the cycle $C_{2^n+l-1} = (v_1, u_1, v_2, v_3, u_2, v_4, v_5, u_3, v_6, v_7, \dots, v_{2^n-3}, u_l, v_{2^n-2}, v_{2^n-1}, v_1)$. Now define a set-indexer g of C_{2^n+l-1} with indexing set $Y = X \cup \{x\}$ as follows: $g(v_i) = f(v_i)$; $1 \leq i \leq 2^n - 1$ and $g(u_j) = f(v_{2j-1}, v_{2j}) \cup \{x\}$; $1 \leq j \leq l$. Then by Theorem 2.4 we have $\gamma(C_{2^n+l-1}) = n + 1$. Now by removing the vertices u_j ; $2 \leq j \leq l$ and joining $v_{2j-1}v_{2j}$ in succession we get the cycles $C_{2^n+l-2}, C_{2^n+l-3}, \dots, C_{2^n}$. Clearly g induces optimal set-indexers for these cycles by Theorem 2.4 and $\gamma(C_k) = n + 1$; $2^n \leq k \leq 2^n + l - 2$ so that these cycles are set-semigraceful. \square

Theorem 3.16 *The set-indexing number of the twing graph obtained from P_{2^n-1} is $n+2$; $n \geq 3$ and hence it is set-semigraceful.*

Proof Let $P_{2^n-1} = (v_1, \dots, v_{2^n-1})$. By Theorem 2.10 we have $\gamma(P_{2^n-1}) = n$. Let f be

an optimal set-indexer of P_{2^n-1} with indexing set X . Let T be a twing graph obtained from P_{2^n-1} by joining each vertex v_i ; $i \in \{2, 3, \dots, 2^n - 2\}$ of P_{2^n-1} to two new vertices say u_i and w_i by pendant edges. Consider the set-indexer g of T with indexing set $Y = X \cup \{x, y\}$ defined as follows: $g(v) = f(v)$ for all $v \in V(P_{2^n-1})$, $g(u_i) = f(u_{i-1}) \cup \{x\}$ and $g(w_i) = f(u_{i-1}) \cup \{y\}$; $2 \leq i \leq 2^n - 2$. Consequently $\gamma(T) \leq n + 2$. But by Theorem 2.4 we have

$$\lceil \log_2(|E(T)| + 1) \rceil = \lceil \log_2(2^n - 2 + 2^n - 3 + 2^n - 3 + 1) \rceil = n + 2 \leq \gamma(T).$$

Thus T is set-semigraceful. \square

§4. Construction of Set-Semigraceful Graphs

In this section we construct more set-semigraceful graphs from given ones through various graph theoretic methods.

Theorem 4.1 *Every set-semigraceful (p, q) -graph G with $\gamma(G) = m$ can be embedded in a set-semigraceful $(2^m, q)$ -graph.*

Proof Let f be a set-semigraceful labeling of G with indexing set X of cardinality $m = \gamma(G)$. Now add $2^m - p$ isolated vertices to G and assign the unassigned subsets of X under f to these vertices in a one to one manner. Clearly the resulting graph is set-semigraceful. \square

Theorem 4.2 *A graph G is set-semigraceful with $\gamma(G) = m$, then every subgraph H of G with $2^{m-1} \leq |E(H)| \leq 2^m - 1$ is also set-semigraceful.*

Proof Since every set-indexer of G is a set-indexer of H , the result follows from Theorem 2.4. \square

Corollary 4.3 *All subgraphs G of the star $K_{1,n}$ is set-semigraceful with the same set-indexing number m if and only if $2^{m-1} \leq |E(G)| \leq 2^m - 1$.*

Proof The proof follows from Theorems 4.2 and 2.9. \square

Theorem 4.4 *If a $(p, p-1)$ -graph G is set-semigraceful, then $G \vee N_{2^n-1}$ is set-semigraceful.*

Proof Let G be set-semigraceful with set-indexing number m . By Theorem 2.4 we have

$$\begin{aligned} \gamma(G \vee N_{2^n-1}) &\geq \lceil \log_2(|E(G \vee N_{2^n-1})| + 1) \rceil \\ &= \lceil \log_2(p - 1 + p(2^n - 1) + 1) \rceil \\ &= \lceil \log_2 p(2^n) \rceil = \lceil \log_2(2^n) + \log_2(p) \rceil \geq n + m. \end{aligned}$$

Let f be a set-semigraceful labeling of G with indexing set X of cardinality m . Consider the set $Y = \{y_1, \dots, y_n\}$ and let $V(N_{2^n-1}) = \{v_1, \dots, v_n\}$. We can find a set-semigraceful labeling say g of $G \vee N_{2^n-1}$ with indexing set $X \cup Y$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and assign the distinct nonempty subsets of Y to the vertices v_1, \dots, v_n in any order. Thus $G \vee N_{2^n-1}$ is set-semigraceful. \square

Remark 4.5 The converse of Theorem 4.4 is not true in general. For example consider the wheel graph $W_5 = C_5 \vee K_1 = (u_1, u_2, \dots, u_5, u_1) \vee \{u\}$. Now assigning the subsets $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, d\}$, $\{a, b, c, d\}$ and ϕ of the set $X = \{a, b, c, d\}$ to the vertices u_1, \dots, u_5 and u in that order we get W_5 as set-semigraceful whereas C_5 is not set-semigraceful by Theorem 2.14.

Corollary 4.6 *The triangular book $K_2 \vee N_{2^n-1}$ is set-semigraceful.*

Proof The proof follows from Theorems 4.4 and 2.10. \square

Theorem 4.7 *The fan graph F_n is set-semigraceful if and only if $n \neq 2^m + 1$; $n \geq 4$.*

Proof If $n-1 \neq 2^m$, by Theorem 2.10 we have P_{n-1} is set-semigraceful so that by Theorem 4.4, $F_n = P_{n-1} \vee K_1$ is set-semigraceful.

Conversely if F_n is set-semigraceful, then by Theorem 3.3, we have

$$\gamma(F_n) = \lceil \log_2(n-1+n-2+1) \rceil = \lceil \log_2(2n-2) \rceil = \lceil \log_2(n-1) \rceil + 1.$$

But by Theorem 2.17 we already have $\gamma(F_n) = \lceil \log_2 n \rceil + 1$. Consequently we must have $n \neq 2^m + 1$. \square

Theorem 4.8 *Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph.*

Proof Let $\{v_1, \dots, v_n\}$ be the vertex set of the given graph G . Now take a new vertex say u and join it with all the vertices of G . Consider the set $X = \{x_1, \dots, x_n\}$. Let $m = 2^n - (|E| + n) - 1$. Take m new vertices u_1, \dots, u_m and join them with u . A set-indexer of the resulting graph G' can be defined as follows: Assign ϕ to u and $\{x_i\}$ to v_i ; $1 \leq i \leq n$. Let $S = \{f(e) : e \in E\} \cup \{\{x_i\} : 1 \leq i \leq n\}$. Note that $|S| = |E| + n$. Now by assigning the m elements of $2^X - (S \cup \phi)$ to the vertices u_1, \dots, u_m in any order we get a set-indexer of G' with X as the indexing set, making G' set-semigraceful. \square

Theorem 4.9 *The splitting graph $S'(G)$ of a set-semigraceful bipartite (p, q) -graph G with $\gamma(G) = m$ and $3q \geq 2^{m+1}$, is set-semigraceful.*

Proof Let f be an optimal set-indexer of G with indexing set X of cardinality m . Let $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{u_1, \dots, u_l\}$ be the partition of $V(G)$, where $n = p - k$. Since G is set-semigraceful with $\gamma(G) = m$, by Theorem 3.3 we have $2^{m-1} \leq q \leq 2^m - 1$. To form the splitting graph $S'(G)$ of G , for each v_i or u_j in G , add a new vertex v'_i or u'_j and add edges joining v'_i or u'_j to all neighbours of v_i or u_j in G respectively. Since $S'(G)$ has $3q$ edges, by Theorem 2.4 we have

$$\gamma(S'(G)) \geq \lceil \log_2(|E(S'(G))| + 1) \rceil = \lceil \log_2(3q + 1) \rceil \geq \lceil \log_2(2^{m+1} + 1) \rceil = m + 2.$$

We can define a set-indexer g of $S'(G)$ with indexing set $Y = X \cup \{x, y\}$ as follows: $g(v) = f(v)$ for all $v \in V(G)$, $g(v'_i) = f(v_i) \cup \{x\}$; $1 \leq i \leq n$ and $g(u'_j) = f(u_j) \cup \{y\}$; $1 \leq j \leq l$. Consequently

$$\gamma(S'(G)) = m + 2 = \lceil \log_2(|E(S'(G))| + 1) \rceil$$

and hence $S'(G)$ is set-semigraceful. \square

Remark 4.10 (1) Even though C_3 is not bipartite, both C_3 and its splitting graph are set-semigraceful.

(2) Splitting graph of a path P_4 is set-semigraceful. But P_4 is not set-semigraceful.

Theorem 4.11 *For any set-graceful graph G , the graph $H; G \cup K_1 \subset H \subseteq G \vee K_1$ is set-semigraceful.*

Proof Let $m = \gamma(G) = \log_2(|E(G)| + 1)$. Then by Theorem 2.4 we have

$$\gamma(H) \geq \lceil \log_2(|E(H)| + 1) \rceil \geq \lceil \log_2(2^m + 1) \rceil = m + 1.$$

Let f be a set-graceful labeling of G with indexing set X . Now we can extend f to a set-indexer g of $G \vee K_1$ with indexing set $Y = X \cup \{x\}$ of cardinality $m + 1$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and $g(v) = \{x\}$ where $\{v\} = V(G)$. Clearly $g(e) = f(e)$ for all $e \in E(G)$ and $g(uv) = g(u) \cup \{x\}$ are all distinct. Then by Theorem 2.5 we have

$$\gamma(H) = m + 1 = \lceil \log_2(|E(H)| + 1) \rceil. \quad \square$$

Corollary 4.12 *The wheel W_{2^n-1} is set-semigraceful.*

Proof The proof follows from Theorems 4.11 and 2.30. \square

Theorem 4.13 *Let G be a set-graceful $(p, p-1)$ -graph, then $G \vee N_m$ is set-semigraceful.*

Proof Let G be set-graceful graph with set-indexing number n . For every m , there exists l such that $2^l \leq m \leq 2^{l+1} - 1$. By Theorem 2.4 we have

$$\begin{aligned} \gamma(G \vee N_m) &\geq \lceil \log_2(|E(G \vee N_m)| + 1) \rceil \\ &= \lceil \log_2(p - 1 + pm + 1) \rceil = \lceil \log_2 p(m + 1) \rceil \\ &= \lceil \log_2(2^n)(m + 1) \rceil = \lceil \log_2(2^n) + \log_2(m + 1) \rceil \geq n + l + 1. \end{aligned}$$

Let f be a set-semigraceful labeling of G with X ; $|X| = n$ as the indexing set. Consider the set $Y = \{y_1, \dots, y_{l+1}\}$ and $V(N_m) = \{v_1, \dots, v_m\}$. Now we can extend f to $G \vee N_m$ by assigning the distinct nonempty subsets of Y to the vertices v_1, \dots, v_m in that order to get a set-indexer of $G \vee N_m$ with indexing set $X \cup Y$. Hence $G \vee N_m$ is set-semigraceful. \square

Corollary 4.14 *$K_{1,2^n-1,m}$ is set-semigraceful.*

Proof The proof follows from Theorems 2.33 and 4.13. \square

Theorem 4.15 *Let G be a $(p, p-1)$ set-graceful graph, then $G \vee K_2$ and $G \vee K_3$ are set-semigraceful.*

Proof Let f be a set-graceful labeling of G with indexing set X of cardinality n . By

Theorem 2.4 we have

$$\begin{aligned}
\gamma(G \vee K_2) &\geq \lceil \log_2(|E(G \vee K_2)| + 1) \rceil = \lceil \log_2(p - 1 + 2p + 1 + 1) \rceil \\
&= \lceil \log_2(3p + 1) \rceil = \left\lceil \log_2 3 + \log_2\left(p + \frac{1}{3}\right) \right\rceil \\
&= \left\lceil \log_2 3 + \log_2\left(2^n + \frac{1}{3}\right) \right\rceil \geq n + 2.
\end{aligned}$$

Consider the set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ and the set-indexer g of $G \vee K_2$ defined by $g(u) = f(u)$ for all $u \in V(G)$, $g(v_1) = \{x_{n+1}\}$ and $g(v_2) = \{x_{n+2}\}$; $v_1, v_2 \in V(K_2)$. Consequently $\gamma(G \vee K_2) = n + 1$ so that it is set-semigraceful.

Let $K_3 = (v_1, v_2, v_3, v_1)$. By Theorem 2.4 we have $\gamma(G \vee K_3) \geq \lceil \log_2(|E(G \vee K_3)| + 1) \rceil = \lceil \log_2(p - 1 + 3p + 3 + 1) \rceil = \lceil \log_2(4p + 3) \rceil = \lceil \log_2(4 \cdot 2^n + 3) \rceil = \lceil \log_2(2^{n+2} + 3) \rceil \geq n + 3$. We can find a set-indexer h of $G \vee K_3$ with indexing set $Z = Y \cup \{x_{n+3}\}$ as follows: Assign $\{x_{n+1}\}$, $\{x_{n+2}\}$ and $\{x_{n+3}\}$ to the vertices of K_3 and $h(u) = f(u)$ for all $u \in V(G)$. Clearly h is a set-semigraceful labeling of $G \vee K_3$. \square

Corollary 4.16 *All double fans $P_n \vee K_2$; $n \neq 2^m$, $m \geq 2$ are set-semigraceful.*

Proof The proof follows from Theorems 4.15 and 2.10. \square

Corollary 4.17 *The graph $K_{1,2^n-1} \vee K_2$ is set-semigraceful.*

Proof The proof follows from Theorems 4.15 and 2.13. \square

Theorem 4.18 *If C_n is set-semigraceful, then the graph $C_n \vee K_2$ is set-semigraceful. Moreover $\gamma(C_n \vee K_2) = m + 2$, where $2^m \leq n \leq 2^m + 2^{m-1} - 2$, $n \geq 7$.*

Proof The proof follows from Theorems 3.15 and 4.15. \square

Theorem 4.19 *Let G be a set-semigraceful (p, q) -graph with $\gamma(G) = m$. If $p \geq 2^{m-1}$, then $G \vee K_1$ is set-semigraceful.*

Proof By Theorem 3.3 we have $2^{m-1} \leq |E(G)| \leq 2^m - 1$. Since $|V| \geq 2^{m-1}$, by Theorem 2.4 we have $\gamma(G \vee K_1) \geq \lceil \log_2(|E| + 1) \rceil = m + 1$. Let f be a set-indexer of G with indexing set X of cardinality $m = \gamma(G)$. Now we can define a set-indexer g of $G \vee K_1$; $V(K_1) = \{v\}$ with indexing set $Y = X \cup \{x\}$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and $g(v) = Y$. This shows that $G \vee K_1$ is set-semigraceful. \square

Corollary 4.20 *If C_m is set-semigraceful, then W_m is also set-semigraceful.*

Proof The proof follows from Theorem 4.19. \square

Corollary 4.21 *W_n is set-semigraceful, where $2^m + 1 \leq n \leq 2^m + 2^{m-1} - 1$.*

proof The proof follows from Theorem 3.15 and Corollary 4.20. \square

Theorem 4.22 *The gear graph of order $2n + 1$ with $2^m - 1 \leq n \leq 2^{m-1} + 2^{m-3}$, $m \geq 3$ is set-semigraceful.*

Proof The proof follows from Theorem 2.5 and Corollary 4.21. \square

Theorem 4.23 *Let G be a set-semigraceful hamiltonian (p, q) -graph with $\gamma(G) = m$ and $p \geq 2^{m-1}$. If G' is a graph obtained from G by joining a pendant vertex to each vertex of G , then G' is set-semigraceful.*

Proof Let $C = (v_1, v_2, \dots, v_n, v_1)$ be a hamiltonian cycle in G . Let f be a set-indexer of G with $\gamma(G) = m$ and X be the corresponding indexing set. Now take n new vertices v'_i ; $1 \leq i \leq n$ and let $G' = G \cup \{v_i v'_i / 1 \leq i \leq n\}$. By Theorem 2.4 we have $\gamma(G') \geq \lceil \log_2(|E(G')| + 1) \rceil = m + 1$. We can define a set-indexer g of G' with indexing set $Y = X \cup \{x\}$ as follows: $g(u) = f(u)$ for all $u \in V(G)$, $g(v'_i) = f(v_i v_{i+1}) \cup \{x\}$; $1 \leq i \leq n$ with $v_{n+1} = v_1$. Clearly G' is set-semigraceful. \square

Corollary 4.24 *If C_n is set-semigraceful, then the sun-graph obtained from C_n is set-semigraceful.*

Proof The proof follows from Theorem 4.23. \square

Corollary 4.25 *The sun-graph of order $2n$; $2^m \leq n \leq 2^m + 2^{m-1} - 2$; $m \geq 3$ is set-semigraceful.*

Proof The proof follows from Theorem 3.15 and Corollary 4.24. \square

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On Generalized m -Power Matrices and Transformations

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Abstract: In this paper, generalized m -power matrices and generalized m -power transformations are defined and studied. First, we give two equivalent characterizations of generalized m -power matrices, and extend the corresponding results about m -idempotent matrices and m -unit-ponent matrices. And then, we also generalize the relative results of generalized m -power matrices to the ones of generalized m -power transformations.

Key Words: Generalized m -power matrix, generalized m -power transformation, equivalent characterization.

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§1. Introduction

The m -idempotent matrices and m -unit-ponent matrices are two typical matrices and have many interesting properties (for example, see [1]-[5]).

A matrix $A \in \mathbb{C}^{n \times n}$ is called an m -idempotent (m -unit-ponent) matrix if there exists positive integer m such that $A^m = A$ ($A^m = I$). Notice that

$$A^m = A \text{ if and only if } \prod_{i=1}^m (A + \varepsilon_i I) = O,$$

where $\varepsilon_1 = 0, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_m$ are the $m-1$ power unit roots,

$$A^m = I \text{ if and only if } \prod_{i=1}^m (A + \varepsilon_i I) = O,$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ are the m -power unit roots. Naturally, we will consider the class of matrices which satisfies that

$$\prod_{i=1}^m (A + \lambda_i I) = O,$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the pairwise different complex numbers.

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For convenience, we call a matrix $A \in \mathbb{C}^{n \times n}$ to be a generalized m -power matrix if it satisfies that $\prod_{i=1}^m (A + \lambda_i I) = O$, where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the pairwise different complex numbers.

In this paper, we firstly study the generalized m -power matrices, and give two equivalent characterizations of such matrices. Consequently, the corresponding results about m -idempotent matrices and m -unit-ponent matrices are generalized. And then, we also define the generalized m -power transformations, and generalize the relative results of generalized m -power matrices to those of generalized m -power transformations.

For terminologies and notations occurred but not mentioned in this paper, the readers are referred to the reference [6].

§2. Generalized m -Power Matrices

In this section, we are going to study some equivalent characterizations of generalized m -power matrices. First, we introduce some lemmas following.

Lemma 2.1([4]) *Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n \times n}$. Then*

$$r\left(\prod_{i=1}^m (A + \lambda_i I)\right) = \sum_{i=1}^m r(A + \lambda_i I) - (m-1)n.$$

Lemma 2.2([4]) *Let $f_1(x), f_2(x), \dots, f_m(x) \in \mathbb{C}[x]$ be pairwise co-prime and $A \in \mathbb{C}^{n \times n}$. Then*

$$\sum_{i=1}^m r(f_i(A)) = (m-1)n + r\left(\prod_{i=1}^m (f_i(A))\right).$$

Lemma 2.3([1]) *Assume that $f(x), g(x) \in \mathbb{C}[x]$, $d(x) = (f(x), g(x))$ and $m(A) = [f(x), g(x)]$. Then for any $A \in \mathbb{C}^{n \times n}$,*

$$r(f(A)) + r(g(A)) = r(d(A)) + r(m(A)).$$

Theorem 2.4 *Let $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{C}$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n \times n}$. Then $\prod_{i=1}^m (A + \lambda_i I) = O$ if and only if $\sum_{i=1}^m r(A + \lambda_i I) = (m-1)n$.*

Proof Assume that $\prod_{i=1}^m (A + \lambda_i I) = O$, by Lemma 2.1, we can immediately get $\sum_{i=1}^m r(A + \lambda_i I) = (m-1)n$.

Assume that $\sum_{i=1}^m r(A + \lambda_i I) = (m-1)n$. Take $f_i(x) = x + \lambda_i$ ($i = 1, 2, \dots, m$), where $\lambda_i \neq \lambda_j$ if $i \neq j$. Clearly, we have $(f_i(x), f_j(x)) = 1$ if $i \neq j$. Now, by Lemma 2.2,

$$\sum_{i=1}^m r(f_i(A)) = (m-1)n + r\left(\prod_{i=1}^m (f_i(A))\right).$$

Also, since $\sum_{i=1}^m r(A + \lambda_i I) = (m-1)n$, we can get $r\left(\prod_{i=1}^m (f_i(A))\right) = 0$, this implies that

$$\prod_{i=1}^m (A + \lambda_i I) = O. \quad \square$$

By Theorem 2.4, we can obtain the following conclusions. Consequently, the corresponding result in [3] is generalized.

Corollary 2.5 *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} \in \mathbb{C}$ be the $m-1$ power unit roots and $A \in \mathbb{C}^{n \times n}$. Then $A^m = A$ if and only if $r(A) + r(A - \varepsilon_1 I) + r(A - \varepsilon_2 I) + \dots + r(A - \varepsilon_{m-1} I) = (m-1)n$.*

Corollary 2.6 *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \in \mathbb{C}$ be the m power unit roots and $A \in \mathbb{C}^{n \times n}$. Then $A^m = I$ if and only if $r(A - \varepsilon_1 I) + r(A - \varepsilon_2 I) + \dots + r(A - \varepsilon_m I) = (m-1)n$.*

Now, we give another equivalent characterizations of the generalized m -power matrices.

Theorem 2.7 *Let $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{C}$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n \times n}$. Then*

$$\prod_{i=1}^m (A + \lambda_i I) = O \text{ if and only if } \sum_{1 \leq i < j \leq m} r((A + \lambda_i I)(A + \lambda_j I)) = \frac{(m-2)(m-1)n}{2}.$$

Proof Assume that $\prod_{i=1}^m (A + \lambda_i I) = O$. Then $r(\prod_{i=1}^m (A + \lambda_i I)) = 0$. Notice that

$$\sum_{1 \leq i < j \leq m} r((A + \lambda_i I)(A + \lambda_j I)) = \sum_{i=1}^{m-1} \sum_{j=i+1}^m r((A + \lambda_i I)(A + \lambda_j I))$$

and by Lemmas 2.2 and 2.3, it is not hard to get that

$$\begin{aligned} \sum_{j=2}^m r((A + \lambda_1 I)(A + \lambda_j I)) &= (m-2) \cdot r(A + \lambda_1 I) + r\left(\prod_{i=1}^m (A + \lambda_i I)\right) \\ &= (m-2) \cdot r(A + \lambda_1 I), \\ \sum_{j=3}^m r((A + \lambda_2 I)(A + \lambda_j I)) + r(A + \lambda_1 I) &= n + (m-3) \cdot r(A + \lambda_2 I), \\ \sum_{j=4}^m r((A + \lambda_3 I)(A + \lambda_j I)) + r(A + \lambda_1 I) + r(A + \lambda_2 I) &= 2n + (m-4) \cdot r(A + \lambda_3 I), \\ &\dots\dots\dots, \\ \sum_{j=m-1}^m r((A + \lambda_{m-2} I)(A + \lambda_j I)) + \sum_{i=1}^{m-3} r(A + \lambda_i I) &= (m-3) \cdot n + r(A + \lambda_{m-2} I), \\ r((A + \lambda_{m-1} I)(A + \lambda_m I)) + \sum_{i=1}^{m-2} r(A + \lambda_i I) &= (m-2) \cdot n. \end{aligned}$$

Thus, we have

$$\sum_{1 \leq i < j \leq m} r((A + \lambda_i I)(A + \lambda_j I)) = \frac{(m-2)(m-1)n}{2}.$$

From the discussions above, we have

$$\sum_{1 \leq i < j \leq m} r((A + \lambda_i I)(A + \lambda_j I)) = \frac{(m-2)(m-1)n}{2} + (m-1) \cdot r\left(\prod_{i=1}^m (A + \lambda_i I)\right).$$

Hence, if

$$\sum_{1 \leq i < j \leq m} r((A + \lambda_i I)(A + \lambda_j I)) = \frac{(m-2)(m-1)n}{2},$$

then

$$r\left(\prod_{i=1}^m (A + \lambda_i I)\right) = 0,$$

i.e.,

$$\prod_{i=1}^m (A + \lambda_i I) = O. \quad \square$$

By Theorem 2.7, we can get corollaries following. Also, the corresponding result in [3] is generalized.

Corollary 2.8 *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} \in \mathbb{C}$ be the $m-1$ power unit roots and $A \in \mathbb{C}^{n \times n}$. Then $A^m = A$ if and only if $r(A(A - \varepsilon_1 I)) + \dots + r(A(A - \varepsilon_{m-1} I)) + r((A - \varepsilon_1 I)(A - \varepsilon_2 I)) + \dots + r((A - \varepsilon_1 I)(A - \varepsilon_{m-1} I)) + \dots + r((A - \varepsilon_{m-2} I)(A - \varepsilon_{m-1} I)) = \frac{(m-2)(m-1)n}{2}$.*

Corollary 2.9 *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \in \mathbb{C}$ be the m power unit roots and $A \in \mathbb{C}^{n \times n}$. Then $A^m = I$ if and only if*

$$\sum_{1 \leq i < j \leq m} r((A + \varepsilon_i I)(A + \varepsilon_j I)) = \frac{(m-2)(m-1)n}{2}.$$

§3. Generalized m -Power Transformations

In this section, analogous with the discussions of the generalized m -power matrices, we will firstly introduce the concepts of generalized m -power linear transformations, and then study some of their properties.

Let V be a n dimensional vector space over a field F and σ a linear transformation on V . We call σ to be a generalized m -power transformation if it satisfies that

$$\prod_{i=1}^m (\sigma + \lambda_i \epsilon) = \theta$$

for pairwise different complex numbers $\lambda_1, \lambda_2, \dots, \lambda_m$, where ϵ is the identical transformation and θ is the null transformation. Especially, σ is called an m -idempotent (m -unit-ponent) transformation if it satisfies that $\sigma^m = \sigma(\sigma^m = \epsilon)$.

From [6], it is known that n dimensional vector space V over a field F is isomorphic to F^n and the linear transformation space $L(V)$ is isomorphic to $F^{n \times n}$. Thus, we can obtain the following results about generalized m -power transformations whose proofs are similar with the corresponding ones in Section 2. And we omit them here.

Theorem 3.1 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is a generalized m -power transformation if and only if*

$$\sum_{i=1}^m \dim \text{Im}(\sigma + \lambda_i \epsilon) = (m-1)n.$$

By Theorem 3.1, we obtain the following conclusions.

Corollary 3.2 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is an m -idempotent transformation if and only if*

$$\dim \text{Im}(A) + \dim \text{Im}(A - \varepsilon_1 I) + \dim \text{Im}(A - \varepsilon_2 I) + \cdots + \dim \text{Im}(A - \varepsilon_{m-1} I) = (m-1)n.$$

Corollary 3.3 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is an m -unit-ponent transformation if and only if*

$$\dim \text{Im}(A - \varepsilon_1 I) + \dim \text{Im}(A - \varepsilon_2 I) + \cdots + \dim \text{Im}(A - \varepsilon_m I) = (m-1)n.$$

Corollary 4.4 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is a generalized m -power transformation if and only if*

$$\sum_{1 \leq i < j \leq m} \dim \text{Im}((\sigma + \lambda_i \epsilon)(\sigma + \lambda_j \epsilon)) = \frac{(m-2)(m-1)n}{2}.$$

Corollary 3.5 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is an m -idempotent transformation if and only if*

$$\begin{aligned} & \dim \text{Im}(\sigma(\sigma - \varepsilon_1 \epsilon)) + \cdots + \dim \text{Im}(\sigma(\sigma - \varepsilon_{m-1} \epsilon)) \\ & + \dim \text{Im}(\sigma - \varepsilon_1 \epsilon)(\sigma - \varepsilon_2 \epsilon) + \cdots + \dim \text{Im}((\sigma - \varepsilon_1 \epsilon)(\sigma - \varepsilon_{m-1} \epsilon)) \\ & + \cdots + \dim \text{Im}((\sigma - \varepsilon_{m-2} \epsilon)(\sigma - \varepsilon_{m-1} \epsilon)) = \frac{(m-2)(m-1)n}{2}. \end{aligned}$$

Corollary 3.6 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V . Then σ is an m -unit-ponent transformation if and only if*

$$\sum_{1 \leq i < j \leq m} \dim \text{Im}((\sigma + \varepsilon_i \epsilon)(\sigma + \varepsilon_j \epsilon)) = \frac{(m-2)(m-1)n}{2}.$$

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Perfect Domination Excellent Trees

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Abstract: A set D of vertices of a graph G is a *perfect dominating set* if every vertex in $V \setminus D$ is adjacent to exactly one vertex in D . In this paper we introduce the concept of *perfect domination excellent graph* as a graph in which every vertex belongs to some perfect dominating set of minimum cardinality. We also provide a constructive characterization of perfect domination excellent trees.

Key Words: Tree, perfect domination, Smarandachely k -dominating set, Smarandachely k -domination number.

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§1. Introduction

Let $G = (V, E)$ be a graph. A set D of vertices is a perfect dominating set if every vertex in $V \setminus D$ is adjacent to exactly one vertex in D . The perfect domination number of G , denoted $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of G . A perfect dominating set of cardinality $\gamma_p(G)$ is called a $\gamma_p(G)$ -set. Generally, a set of vertices S in a graph G is said to be a *Smarandachely k -dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k -domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k -dominating set of G . Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is nothing but the domination number of G and denoted by $\gamma(G)$. Domination and its parameters are well studied in graph theory. For a survey on this subject one can go through the two books by Haynes et al [3,4].

Sumner [7] defined a graph to be γ -*excellent* if every vertex is in some minimum dominating set. Also, he has characterized γ -*excellent* trees. Similar to this concept, Fricke et al [2] defined a graph to be i -*excellent* if every vertex is in some minimum independent dominating set. The i -excellent trees have been characterized by Haynes et al [5]. Fricke et al [2] defined γ_t -*excellent* graph as a graph in which every vertex is in some minimum total dominating set. The γ_t -excellent trees have been characterized by Henning [6].

In this paper we introduce the concept of γ_p -excellent graph. Also, we provide a constructive characterization of perfect domination excellent trees.

We define the perfect domination number of G relative to the vertex u , denoted $\gamma_p^u(G)$, as the minimum cardinality of a perfect dominating set of G that contains u . We call a perfect dominating set of cardinality $\gamma_p^u(G)$ containing u to be a $\gamma_p^u(G)$ -set. We define a graph G to be

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γ_p – excellent if $\gamma_p^u(G) = \gamma_p(G)$ for every vertex u of G .

All graphs considered in this paper are finite and simple. For definitions and notations not given here see [4]. A tree is an acyclic connected graph. A *leaf* of a tree is a vertex of degree 1. A *support vertex* is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex that is adjacent to more than one leaf.

§2. Perfect Domination Excellent Graph

Proposition 2.1 *A path P_n is γ_p – excellent if and only if $n = 2$ or $n \equiv 1(\text{mod } 3)$.*

Proof It is easy to see that the paths P_2 and P_n for $n \equiv 1(\text{mod } 3)$ are γ_p -excellent. Let $P_n, n \geq 3$, be a γ_p -excellent path and suppose that $n \equiv 0, 2(\text{mod } 3)$. If $n \equiv 0(\text{mod } 3)$, then P_n has a unique γ_p -set, which does not include all the vertices. If $n \equiv 2(\text{mod } 3)$, then no γ_p -set of P_n contains the third vertex on the path. \square

Proposition 2.2 *Every graph is an induced subgraph of a γ_p -excellent graph.*

Proof Consider a graph H and let $G = HoK_1$, the 1-corona of a graph H . Every vertex in $V(H)$ is now a support vertex in G . Therefore, $V(H)$ is a γ_p -set of G . As well, the set of end vertices in G is a γ_p -set. Hence every vertex in $V(G)$ is in some γ_p -set and G is γ_p -excellent. Since H is an induced subgraph of G , the result follows. \square

§3. Characterization of Trees

We now provide a constructive characterization of perfect domination excellent trees. We accomplish this by defining a family of labelled trees as defined in [1].

Let $\mathcal{F} = \{T_n\}_{n \geq 1}$ be the family of trees constructed inductively such that T_1 is a path P_4 and $T_n = T$, a tree. If $n \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the two operations $\mathcal{F}_1, \mathcal{F}_2$ for $i = 1, 2, \dots, n-1$. Then we say that T has length n in \mathcal{F} .

We define the status of a vertex v , denoted $\text{sta}(v)$ to be A or B . Initially if $T_1 = P_4$, then $\text{sta}(v) = A$ if v is a support vertex and $\text{sta}(v) = B$, if v is a leaf. Once a vertex is assigned a status, this status remains unchanged as the tree is constructed.

Operation \mathcal{F}_1 Assume $y \in T_n$ and $\text{sta}(y) = A$. The tree T_{n+1} is obtained from T_n by adding a path x, w and the edge xy . Let $\text{sta}(x) = A$ and $\text{sta}(w) = B$.

Operation \mathcal{F}_2 Assume $y \in T_n$ and $\text{sta}(y) = B$. The tree T_{n+1} is obtained from T_n by adding a path x, w, v and the edge xy . Let $\text{sta}(x) = \text{sta}(w) = A$ and $\text{sta}(v) = B$.

\mathcal{F} is closed under the two operations \mathcal{F}_1 and \mathcal{F}_2 . For $T \in \mathcal{F}$, let $A(T)$ and $B(T)$ be the sets of vertices of status A and B respectively. We have the following observation, which follow from the construction of \mathcal{F} .

Observation 3.1 Let $T \in \mathcal{F}$ and $v \in V(T)$

1. If $\text{sta}(v) = A$, then v is adjacent to exactly one vertex of $B(T)$ and at least one vertex of $A(T)$.
2. If $\text{sta}(v) = B$, then $N(v)$ is a subset of $A(T)$.
3. If v is a support vertex, then $\text{sta}(v) = A$.
4. If v is a leaf, then $\text{sta}(v) = B$.
5. $|A(T)| \geq |B(T)|$
6. Distance between any two vertices in $B(T)$ is at least three.

Lemma 3.2 *If $T \in \mathcal{F}$, then $B(T)$ is a $\gamma_p(T)$ -set. Moreover if T is obtained from $T' \in \mathcal{F}$ using operation \mathcal{F}_1 or \mathcal{F}_2 , then $\gamma_p(T) = \gamma_p(T') + 1$.*

Proof By Observation 3.1, it is clear that $B(T)$ is a perfect dominating set. Now we prove that, $B(T)$ is a $\gamma_p(T)$ -set. We proceed by induction on the length n of the sequence of trees needed to construct the tree T . Suppose $n = 1$, then $T = P_4$, belongs to \mathcal{F} . Let the vertices of P_4 be labeled as a, b, c, d . Then, $B(P_4) = \{a, d\}$ and is a $\gamma_p(P_4)$ -set. This establishes the base case. Assume then that the result holds for all trees in \mathcal{F} that can be constructed from a sequence of fewer than n trees where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence T_1, T_2, \dots, T_n of n trees, where $T' = T_{n-1}$ and $T = T_n$. By our inductive hypothesis $B(T')$ is a $\gamma_p(T')$ -set.

We now consider two possibilities depending on whether T is obtained from T' by operation \mathcal{F}_1 or \mathcal{F}_2 .

Case 1 T is obtained from T' by operation \mathcal{F}_1 .

Suppose T is obtained from T' by adding a path y, x, w of length 2 to the attacher vertex $y \in V(T')$. Any $\gamma_p(T')$ -set can be extended to a $\gamma_p(T)$ -set by adding to it the vertex w , which is of status B . Hence $B(T) = B(T') \cup \{w\}$ is a $\gamma_p(T)$ -set.

Case 2 T is obtained from T' by operation \mathcal{F}_2 .

The proof is very similar to Case 1.

If T is obtained from $T' \in \mathcal{F}$ using operation \mathcal{F}_1 or \mathcal{F}_2 , then T can have exactly one more vertex with status B than T' . Since $\gamma_p(T) = |B(T)|$ and $\gamma_p(T') = |B(T')|$, it follows that $\gamma_p(T) = \gamma_p(T') + 1$. \square

Lemma 3.3 *If $T \in \mathcal{F}$ have length n , then T is a γ_p -excellent tree.*

Proof Since T has length n in \mathcal{F} , T can be obtained from a sequence T_1, T_2, \dots, T_n of trees such that T_1 is a path P_4 and $T_n = T$, a tree. If $n \geq 2$, T_{i+1} can be obtained from T_i by one of the two operations $\mathcal{F}_1, \mathcal{F}_2$ for $i = 1, 2, \dots, n-1$. To prove the desired result, we proceed by induction on the length n of the sequence of trees needed to construct the tree T .

If $n = 1$, then $T = P_4$ and therefore, T is γ_p -excellent. Hence the lemma is true for the base case.

Assume that the result holds for all trees in \mathcal{F} of length less than n , where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence T_1, T_2, \dots, T_n of n trees. For notational convenience, we denote T_{n-1} by T' . We now consider two possibilities depending on whether T is obtained from T' by operation \mathcal{F}_1 or \mathcal{F}_2 .

Case 1 T is obtained from T' by operation \mathcal{F}_1 .

By Lemma 3.2, $\gamma_p(T) = \gamma_p(T') + 1$. Let u be an arbitrary element of $V(T)$.

Subcase 1.1 $u \in V(T')$.

Since T' is γ_p -excellent, $\gamma_p^u(T') = \gamma_p(T')$. Now any $\gamma_p^u(T')$ -set can be extended to a perfect dominating set of T by adding either x or w and so $\gamma_p^u(T) \leq \gamma_p^u(T') + 1 = \gamma_p(T') + 1 = \gamma_p(T)$.

Subcase 1.2 $u \in V(T) \setminus V(T')$.

Any $\gamma_p^y(T')$ -set can be extended to a perfect dominating set of T by adding the vertex w and so $\gamma_p^u(T) \leq \gamma_p^y(T') + 1 = \gamma_p(T') + 1 = \gamma_p(T)$.

Consequently, we have $\gamma_p^u(T) = \gamma_p(T)$ for any arbitrary vertex u of T . Hence T is γ_p -excellent.

Case 2 T is obtained from T' by operation \mathcal{F}_2 .

The proof is very similar to Case 1. □

Proposition 3.4 *If T is a tree obtained from a tree T' by adding a path x, w or a path x, w, v and an edge joining x to the vertex y of T' , then $\gamma_p(T) = \gamma_p(T') + 1$.*

Proof Suppose T is a tree obtained from a tree T' by adding a path x, w and an edge joining x to the vertex y of T' , then any $\gamma_p(T')$ -set can be extended to a perfect dominating set of T by adding x or w and so $\gamma_p(T) \leq \gamma_p(T') + 1$. Now let S be a $\gamma_p(T)$ -set and let $S' = S \cap V(T')$. Then S' is a perfect dominating set of T' . Hence, $\gamma_p(T') \leq |S'| \leq |S| - 1 = \gamma_p(T) - 1$. Thus, $\gamma_p(T) \geq \gamma_p(T') + 1$. Hence $\gamma_p(T) = \gamma_p(T') + 1$. The other case can be proved on the same lines. □

Theorem 3.5 *A tree T of order $n \geq 4$ is γ_p -excellent if and only if $T \in \mathcal{F}$.*

Proof By Lemma 3.3, it is sufficient to prove that the condition is necessary. We proceed by induction on the order n of a γ_p -excellent tree T . For $n = 4$, $T = P_4$ is γ_p -excellent and also it belongs to the family \mathcal{F} . Assume that $n \geq 5$ and all γ_p -excellent trees with order less than n belong to \mathcal{F} . Let T be a γ_p -excellent tree of order n . Let $P : v_1, v_2, \dots, v_k$ be a longest path in T . Obviously $\deg(v_1) = \deg(v_k) = 1$ and $\deg(v_2) = \deg(v_{k-1}) = 2$ and $k \geq 5$. We consider two possibilities.

Case 1 v_3 is a support vertex.

Let $T' = T \setminus \{v_1, v_2\}$. We prove that T' is γ_p -excellent, that is for any $u \in T'$, $\gamma_p^u(T') = \gamma_p(T')$. Since $u \in T' \subset T$ and T is γ_p -excellent, there exists a $\gamma_p^u(T)$ -set such that $\gamma_p^u(T) = \gamma_p(T)$. Let S be a $\gamma_p^u(T)$ -set and $S' = S \cap V(T')$. Then S' is a perfect dominating set of T' . Also, $|S'| \leq |S| - 1 = \gamma_p(T) - 1 = \gamma_p(T')$, by Proposition 3.4. Since $u \in S'$, S' is a $\gamma_p^u(T')$ -set

such that $\gamma_p^u(T') = \gamma_p(T')$. Thus T' is γ_p -excellent. Hence by the inductive hypothesis $T' \in \mathcal{F}$, since $|V(T')| < |V(T)|$. The $sta(v_3) = A$ in T' , because v_3 is a support vertex. Thus, T is obtained from $T' \in \mathcal{F}$ by the operation \mathcal{F}_1 . Hence $T \in \mathcal{F}$ as desired.

Case 2 v_3 is not a support vertex.

Let $T' = T \setminus \{v_1, v_2, v_3\}$. As in Case 1, we can prove that T' is γ_p -excellent. Since $|V(T')| < |V(T)|$, $T' \in \mathcal{F}$ by the inductive hypothesis.

If v_4 is a support vertex or has a neighbor which is a support vertex then v_3 is present in none of the γ_p -sets. So, T cannot be γ_p -excellent. Hence either $\deg(v_4) = 2$ and v_4 is a leaf of T' so that $v_4 \in B(T')$ by Observation 3.1 or $\deg(v_4) \geq 3$ and all the neighbors of v_4 in $T' \setminus \{v_5\}$ are at distance exactly 2 from a leaf of T' . Hence all the neighbors of v_4 in $T' \setminus \{v_5\}$ are in $A(T')$ by Observation 3.1, and have no neighbors in $B(T')$ except v_4 . Hence $v_4 \in B(T')$, again by Observation 3.1. Thus, T can be obtained from T' by the operation \mathcal{F}_2 . Hence $T \in \mathcal{F}$. \square

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On (k, d) -Maximum Indexable Graphs and (k, d) -Maximum Arithmetic Graphs

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Abstract: A (n, m) graph G is said to be (k, d) maximum indexable graph, if its vertices can be assigned distinct integers $0, 1, 2, \dots, n-1$, so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices and maximum of them can be arranged in the arithmetic progression $k, k+1, k+2d, \dots, k+(m-1)d$ and also a (n, m) graph G is said to be (k, d) maximum arithmetic graph, if its vertices can be assigned distinct non negative integers so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices and maximum of them can be arranged in the arithmetic progression $k, k+1, k+2d, \dots, k+(m-1)d$. The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of G . In this paper we introduce some families of graphs which are (k, d) - maximum indexable and (k, d) -maximum arithmetic and also compute energies of some of them.

Key Words: Graph labeling, indexable graphs, (k, d) -Maximum indexable graphs, (k, d) -maximum arithmetic graphs.

AMS(2010): 05C78, 05C50, 58C40

§1. Introduction

Let $G = (V, E)$ be a (n, m) graph and let its vertex set be $V(G) = \{v_1, v_2, \dots, v_n\}$. We assume that G is a finite, undirected, connected graph without loops or multiple edges. Graph labelings where the vertices are assigned values subject to certain conditions are interesting problems and have been motivated by practical problems. Applications of graph labeling have been found in X -ray, crystallography, Coding theory, Radar, Circuit design, Astronomy and communication design.

Given a graph $G = (V, E)$, the set N of non-negative integers, a subset A of N of non-negative integers, a set A of N and a Commutative binary operation $*$: $N \times N \longrightarrow N$ every vertex function $f : V(G) \longrightarrow A$ induces an edge function $f^* : E(G) \longrightarrow N$ such that $f^*(uv) = *(f(u), f(v)) = f(u) * f(v)$, $\forall uv \in E(G)$. We denote such induced map f^* of f by f^{max} .

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Acharya and Hedge [1,2] have introduced the concept of indexable and arithmetic graph labelings. Recently present author [7] has introduced the concept of maximum indexable graphs. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i, j) -entry is equal to unity if the vertices v_i and v_j are adjacent, and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G , and are studied within the Spectral Graph Theory [3]. The energy of the graph G is defined as $E = E(G) = \sum_{i=1}^n |\lambda_i|$. The graph energy is an invariant much such studied in mathematical and mathema-tico-chemical literature; for details see [4,5,6], In this paper we introduce some families of graphs which are (k, d) - maximum indexable and (k, d) -maximum arithmetic and also compute energies of some of them.

Definition 1.1 A graph $G = (V, E)$ is said to be (k, d) - maximum indexable graph if it admits a (k, d) - indexer, namely an injective function $f : V(G) \longrightarrow \{0, 1, \dots, n-1\}$ such that $f(u) + f(v) + \max\{f(u), f(v)\} = f^{\max}(uv) \in f^{\max}(G) = \{f^{\max}(uv) : \forall uv \in E(G)\} = \{k, k+d, k+2d, \dots, k+(m-1)d\}$, for every $uv \in E$.

Example 1.2 The graph $K_{2,3}$ is a $(4, 1)$ -maximum indexable graph.

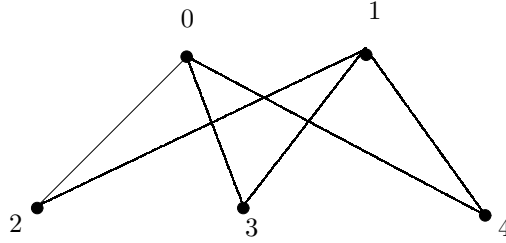


Figure 1

In this example,

$$f^{\max}(G) = \{4, 5, 6, 7, 8, 9\}.$$

Lemma 1.3 Let f be any (k, d) -maximum indexable labeling of G . Then

$$2 \leq k \leq 3n - md + (d - 4).$$

Proof Since $f^{\max}(G) \subseteq \{2, 3, \dots, 3n-4\}$, the largest edge label is at most $3n-4$. Hence k must be less than or equal to $3n-4 - (m-1)d$. Since the edge values are in the set $\{k, k+d, k+2d, \dots, k+(m-1)d\}$, we have

$$2 \leq k \leq 3n - md + (d - 4).$$

□

Theorem 1.4 *The star $K_{1,n}$ is (k, d) -maximum indexable if and only if $(k, d) = (2, 2)$ or $(2n, 1)$. Furthermore, there are exactly $n + 1$ maximum indexable labelings of $K_{1,n}$ of which only two are (k, d) -maximum indexable up to isomorphism.*

Proof By assigning the value 0 to the central vertex and $1, 2, 3, \dots, n$ to the pendent vertices we get $(2, 2)$ -maximum indexable graph, since $f^{\max}(G) = \{2, 4, 6, \dots, 2n\}$. By assigning the value n to the root vertex and $0, 1, 2, \dots, n - 1$ to the pendent vertices, one can see that $f^{\max}(G) = \{2n, 2n + 1, 2n + 3, \dots, 3n - 1\}$, which shows that G is $(2n, 1)$ -maximum indexable graph.

Conversely, if we assign the value c ($0 < c < n$) to the central vertex and $0, 1, 2, \dots, c - 1, c + 1, \dots, n$ to the pendent vertices, we obtain $f^{\max}(K_{1,n}) = \{2c, 2c + 1, \dots, 3c - 1, 3c + 2, \dots, 2n + c\}$ which is not an arithmetic progressive, i.e., $K_{1,n}$ is not a (k, d) -maximum indexable.

For the second part, note that $K_{1,n}$ is of order $n + 1$ and size n . Since there are $n + 1$ vertices and $n + 1$ numbers (from 0 to n), each vertex of $K_{1,n}$ can be labeled in $n + 1$ different ways. Observe that the root vertex is adjacent with all the pendent vertices. So if the root vertex is labeled using $n + 1$ numbers and the pendent vertices by the remaining numbers, definitely the sum of the labels of each pendent vertex with the label of root vertex, will be distinct in all $n + 1$ Maximum Indexable labelings of $K_{1,n}$. It follows from first part that, out of $n + 1$ maximum indexable labelings of $K_{1,n}$, only two are (k, d) -maximum indexable. This completes the proof. \square

Corollary 1.5 *The graph $G = K_{1,n} \cup K_{1,n}$, $n \geq 1$ is (k, d) -maximum indexable graph and its energy is equal to $4\sqrt{n - 1}$.*

Proof Let the graph G be $K_{1,n} \cup K_{1,n}$, $n \geq 1$. Denote

$$V(G) = \{u_1, v_{1j} : 1 \leq j \leq n\} \cup \{u_2, v_{2j} : 1 \leq j \leq n\},$$

$$E(G) = \{u_1 v_{1j} : 1 \leq j \leq n\} \cup \{u_2 v_{2j} : 1 \leq j \leq n\}.$$

Define a function $f : V(G) \longrightarrow \{0, 1, \dots, 2n + 1\}$ by

$$f(u_1) = 2n, \quad f(u_2) = 2n - 1$$

$$f(v_{1j}) = (2j - 1) \quad \text{and} \quad f(v_{2j}) = 2(j - 1),$$

for $j = 1, 2, \dots, n$. Thus

$$f^{\max}(K_{1,n} \cup K_{1,n}) = \{4n + 1, 4n + 2, 4n + 3, \dots, 6n - 2, 6n - 1, 6n\}.$$

Thus $K_{1,n} \cup K_{1,n}$ is $(4n + 1, 1)$ -maximum indexable graph. The eigenvalues of $K_{1,n} \cup K_{1,n}$ are $\pm\sqrt{n - 1}, \pm\sqrt{n - 1}, \underbrace{0, 0, \dots, 0}_{2n - 4 \text{ times}}$. Hence $E(K_{1,n} \cup K_{1,n}) = 4\sqrt{n - 1}$. \square

Theorem 1.6 *For any integer $m \geq 2$, the linear forest $F = nP_3 \cup mP_2$ is a $(2m + 2n, 3)$ maximum indexable graph and its energy is $2(n\sqrt{2} + m)$.*

Proof Let $F = nP_3 \cup mP_2$ be a linear forest and $V(F) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{x_j, y_j : 1 \leq j \leq m\}$. Then $|V(F)| = 3n + 2m$ and $|E(F)| = 2n + m$. Define $f : V(F) \rightarrow \{0, 1, \dots, 2m + 3n - 1\}$ by

$$f(y) = \begin{cases} i - 1, & y = u_i, \quad 1 \leq i \leq n, \\ m + n + (i - 1), & y = v_i, \quad 1 \leq i \leq n, \\ j + n - 1, & y = x_j, \quad 1 \leq j \leq m, \\ 2n + m + (j - 1), & y = y_j, \quad 1 \leq j \leq m, \\ 2(n + m) + (i - 1), & y = w_i, \quad 1 \leq i \leq n. \end{cases}$$

Then

$$\begin{aligned} f^{\max}(F) = & \{2(n + m), 2(n + m) + 3, 2(m + n) + 6 \dots, 2(m + n) + \\ & + 3(n - 1), 5n + 2m, \dots, 5n + 5m - 3, 5n + 5m, 8n + 5m - 3\}. \end{aligned}$$

Therefore $nP_3 \cup mP_2$ is a $(2(n + m), 3)$ -maximum indexable graph.

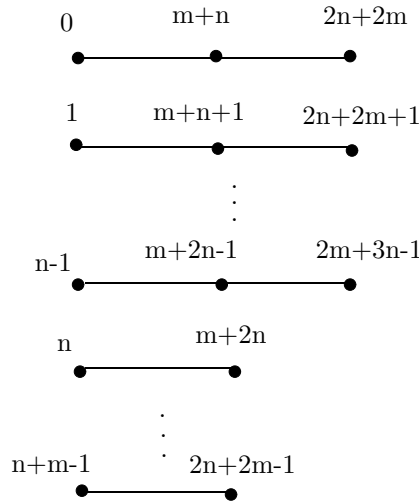


Figure 2

The eigenvalues of $nP_3 \cup mP_2$ are $\underbrace{1, -1}_{m \text{ times}}, \underbrace{0, \sqrt{2}, -\sqrt{2}}_{n \text{ times}}$. Hence its energy is equal to $E(nP_3 \cup mP_2) = 2(n\sqrt{2} + m)$. \square

Corollary 1.7 For any integer $m \geq 2$, the linear forest $F = P_3 \cup mP_2$ is a $(2m+2, 3)$ -maximum indexable graph.

Proof Putting $n = 1$ in the above theorem we get the result. \square

§2. (k, d) – Maximum Arithmetic Graphs

We begin with the definition of (k, d) –maximum arithmetic graphs.

Definition 2.1 Let N be the set of all non negative integers. For a non negative integer k and positive integer d , a (n, m) graph $G = (V, E)$, a (k, d) –maximum arithmetic labeling is an injective mapping $f : V(G) \longrightarrow N$, where the induced edge function $f^{max} : E(G) \longrightarrow \{k, k + d, k + 2d, \dots, k + (m - 1)d\}$ is also injective. If a graph G admits such a labeling then the graph G is called (k, d) –maximum arithmetic.

Example 2.2 Let S_n^3 be the graph obtained from the star graph with n vertices by adding an edge. S_6^3 is an example of $(4, 2)$ –maximum arithmetic graph.

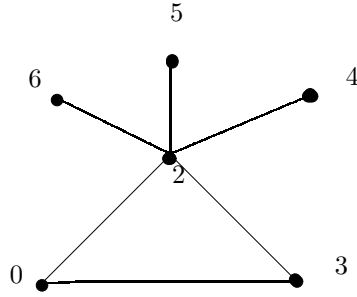


Figure 3 S_6^3

Here

$$f^{max}(S_6^3) = \{4, 6, 8, 10, 12, 14\}.$$

Definition 2.3 Let P_n be the path on n vertices and its vertices be ordered successively as x_1, x_2, \dots, x_n . P_n^l is the graph obtained from P_n by attaching exactly one pendent edge to each of the vertices x_1, x_2, \dots, x_l .

Theorem 2.4 P_n^l is (k, d) –maximum indexable graph.

Proof We consider two cases.

Case 1 If n is odd. In this case we label the vertices of P_n^l as shown in the Figure . The value of edges can be written as arithmetic sequence $\{5, 8, \dots, (3n - 1), (3n + 2), (3n + 5), \dots, (3n + 3l - 1)\}$. It is clear that P_n^l is $(5, 3)$ –maximum arithmetic for any odd n .

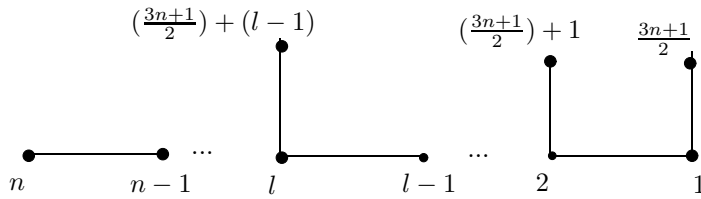


Figure 4

Case 2 If n is even. In this case we label the vertices of P_n^l as shown in the Figure 5. The value of edges can be written as arithmetic sequence $\{7, 13, \dots, (6n-5), (6n+1), (6n+7), (6n+13), \dots, (6n+6l-5)\}$. It is clear that P_n^l is $(7, 6)$ -maximum arithmetic for any even n . \square

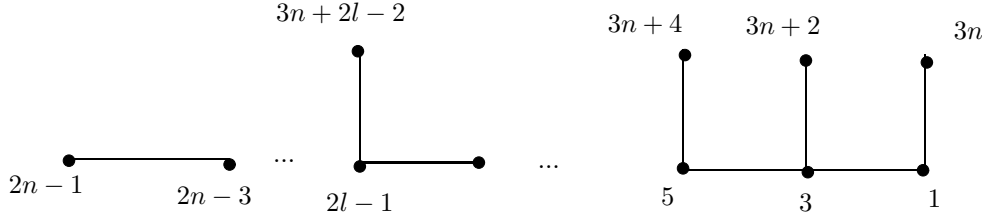


Figure 5

Theorem 2.5 The graph $G = K_2 \nabla \overline{K_{n-2}}$ for $n \geq 3$ is a $(10, 2)$ -maximum arithmetic graph.

Proof Let the vertices of K_2 be v_1 and v_2 and those of $\overline{K_{n-2}}$ be v_3, v_4, \dots, v_n . By letting $f(v_i) = 2i$ for $i = 1, 2$ and $f(v_i) = 2i - 1$, $3 \leq i \leq n$, we can easily arrange the values of edges of $G = K_2 \nabla \overline{K_{n-2}}$ in an increasing sequence $\{10, 12, 14, \dots, 4n, 4n+2\}$. \square

Lemma 2.6 The complete tripartite graph $G = K_{1,2,n}$ is a $(7, 1)$ -maximum arithmetic graph.

Proof Let the tripartite A, B, C be $A = \{w\}$, $B = \{v_1, v_2\}$ and $C = \{v_3, v_4, \dots, v_n\}$. Define the map $f : A \cup B \cup C \longrightarrow N$ by

$$\begin{aligned} f(v_i) &= i, \quad 1 \leq i \leq n+2 \\ f(w) &= n+3. \end{aligned}$$

Then one can see that $G = K_{1,2,n}$ is $(7, 1)$ -maximum arithmetic. In fact $f^{\max}(K_{1,2,n}) = \{7, 8, 9, 10, \dots, 2n+5, 2n+6, 2n+7, \dots, 3n+8\}$. \square

Theorem 2.7 For positive integers m and n , the graph $G = mK_{1,n}$ is a $(2mn + m + 2, 1)$ -maximum arithmetic graph and its energy is $2m\sqrt{n-1}$.

Proof Let the graph G be the disjoint union of m stars. Denote $V(G) = \{v_i : 1 \leq i \leq m\} \cup \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{v_i u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Define $f : V(G) \longrightarrow N$ by

$$f(x) = \begin{cases} m(n+1) - (i-1) & \text{if } x = v_i \quad 1 \leq i \leq m \\ (j-1)m + i & \text{if } x = u_{i,j} \quad 1 \leq i \leq m, 1 \leq j \leq n. \end{cases}$$

Then

$$f^{\max}(G) = \{2mn + m + 2, 2mn + m + 3, \dots, 3mn + m + 1\}.$$

Therefore $mK_{1,n}$ is a $(2mn + m + 2, 1)$ -maximum arithmetic graph.

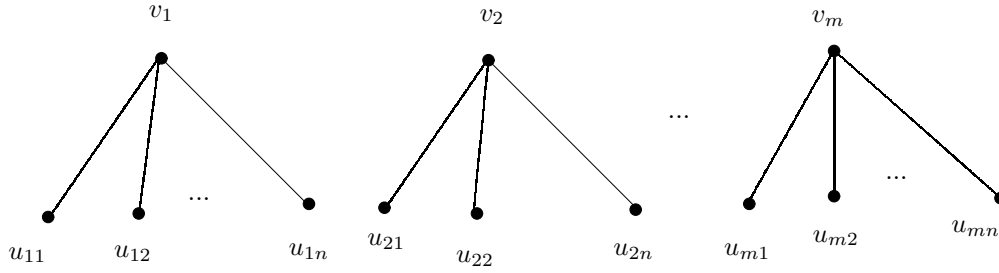


Figure 6

Also its eigenvalues are $\pm\sqrt{n-1}$ (m times). Hence the energy of $mK_{1,n}$ is

$$E(mK_{1,n}) = 2m\sqrt{n-1}.$$

□

§3. Algorithm

In this section we present an algorithm which gives a method to constructs a maximum (k, d) -maximum indexable graphs.

MATRIX-LABELING ($Vlist, Vsize$)

// $Vlist$ is list of the vertices, $Vsize$ is number of vertices

$Elist = \text{Empty}();$

for $j < -0$ to $(Vsize - 1)$

do for $i < -0$ to $(Vlist - 1)$ && $i! = j$

$X = Vlist[i] + Vlist[j] + \text{Max} \{Vlist[i], Vlist[j]\};$

if ($\text{Search}(Elist, X) = \text{False}$) //X does not exist in $Elist$

then ADD ($Elist, X$);

end if

end for

end for

SORT-INCREASINGLY ($Elist$);

Flag= 0;

For $j < -0$ to $(Esize - 1)$

do if ($Elist[j + 1]! = Elist[j] + d$)

then Flag= 1;

end if

end for

if (Flag== 0)

```

then PRINT ("This graph is a  $(k, d)$ –Max indexable graph");
else if (Flag= 1)
    then PRINT ("This graph is not  $(k, d)$ –Max indexable graph");
end if
end MATRIX-LABELING( $Vlist, Vsize$ )

```

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Total Dominator Colorings in Paths

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Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in paths. Unless otherwise specified, n denotes an integer greater than or equal to 2.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k -domination coloring, Smarandachely k -dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [2].

Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v . The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$. A subset S of V is called a dominating (total dominating) set if every vertex in $V - S$ (V) is adjacent to some vertex in S . A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of S is a dominating (total dominating) set of G . The domination number γ (total domination number γ_t) is the minimum cardinality taken over all minimal dominating (total dominating) sets of G . A γ -set (γ_t -set) is any minimal dominating (total dominating) set with cardinality γ (γ_t).

A proper coloring of G is an assignment of colors to the vertices of G , such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \dots, u_p\}$ and $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_n\}$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in \mathcal{C}$ such that $C_i = \{u\}$.

Let G be a graph without isolated vertices. A *total dominator coloring* of a graph G is

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a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the *total dominator chromatic number* of G and is denoted by $\chi_{td}(G)$. Generally, for an integer $k \geq 1$, a *Smarandachely k -dominator coloring* of G is a proper coloring on G such that every vertex in the graph G properly dominates a k color classes and the smallest number of colors for which there exists a Smarandachely k -dominator coloring of G is called the *Smarandachely k -dominator chromatic number* of G , denoted by $\chi_{td}^S(G)$. Clearly, if $k = 1$, such a Smarandachely 1-dominator coloring and Smarandachely 1-dominator chromatic number are nothing but the total dominator coloring and total dominator chromatic number of G .

In this paper we determine total dominator chromatic number in paths.

Throughout this paper, we use the following notations.

Notation 1.1 Usually, the vertices of P_n are denoted by u_1, u_2, \dots, u_n in order. We also denote a vertex $u_i \in V(P_n)$ with $i > \lceil \frac{n}{2} \rceil$ by $u_{i-(n+1)}$. For example, u_{n-1} by u_{-2} . This helps us to visualize the position of the vertex more clearly.

Notation 1.2 For $i < j$, we use the notation $\langle [i, j] \rangle$ for the subpath induced by $\langle u_i, u_{i+1}, \dots, u_j \rangle$. For a given coloring C of P_n , $C|_{\langle [i, j] \rangle}$ refers to the coloring C restricted to $\langle [i, j] \rangle$.

We have the following theorem from [1].

Theorem 1.3 For any graph G with $\delta(G) \geq 1$, $\max\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G)$.

Definition 1.4 We know from Theorem 1.3 that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n) + 1, \gamma_t(P_n) + 2\}$. We call the integer n , good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1, \chi_{td}(P_n) = \gamma_t(P_n)$).

§2. Determination of $\chi_{td}(P_n)$

First, we note the values of $\chi_{td}(P_n)$ for small n . Some of these values are computed in Theorems 2.7, 2.8 and the remaining can be computed similarly.

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
2	2	2
3	2	2
4	2	3
5	3	4
6	4	4

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
7	4	5
8	4	6
9	5	6
10	6	7

Thus $n = 2, 3, 6$ are very bad integers and we shall show that these are the only bad integers. First, we prove a result which shows that for large values of n , the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n \bmod 4$ [More precisely, if n is good, $m > n$ and $m \equiv n \pmod{4}$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 2.1 Let $1 < i < n$ and let C be a td-coloring of P_n . Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C| \langle [i+1, n] \rangle$ is also a td-coloring. This fact is used extensively in this paper.

Lemma 2.2 $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$.

Proof For $2 \leq n \leq 5$, this is directly verified from the table. We may assume $n \geq 6$. Let $u_1, u_2, u_3, \dots, u_{n+4}$ be the vertices of P_{n+4} in order. Let C be a minimal td-coloring of P_{n+4} . Clearly, u_2 and u_{-2} are non-repeated colors. First suppose u_4 is a repeated color. Then $C| \langle [5, n+4] \rangle$ is a td-coloring of P_n . Further, $C| \langle [1, 4] \rangle$ contains at least two color classes of C . Thus $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$. Similarly the result follows if u_{-4} is a repeated color. Thus we may assume u_4 and u_{-4} are non-repeated colors. But the $C| \langle [3, n+2] \rangle$ is a td-coloring and since u_2 and u_{-2} are non-repeated colors, we have in this case also $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$. \square

Corollary 2.3 If for any n , $\chi_{td}(P_n) = \gamma_t(P_n) + 2$, $\chi_{td}(P_m) = \gamma_t(P_m) + 2$, for all $m > n$ with $m \equiv n \pmod{4}$.

Proof By Lemma 2.2, $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2 = \gamma_t(P_n) + 2 + 2 = \gamma_t(P_{n+4}) + 2$. \square

Corollary 2.4 No integer $n \geq 7$ is a very bad integer.

Proof For $n = 7, 8, 9, 10$, this is verified from the table. The result then follows from the Lemma 2.2. \square

Corollary 2.5 The integers 2, 3, 6 are the only very bad integers.

Next, we show that $n = 8, 13, 15, 22$ are good integers. In fact, we determine $\chi_{td}(P_n)$ for small integers and also all possible minimum td-colorings for such paths. These ideas are used more strongly in determination of $\chi_{td}(P_n)$ for $n = 8, 13, 15, 22$.

Definition 2.6 Two td-colorings C_1 and C_2 of a given graph G are said to be equivalent if there exists an automorphism $f : G \rightarrow G$ such that $C_2(v) = C_1(f(v))$ for all vertices v of G . This is clearly an equivalence relation on the set of td-colorings of G .

Theorem 2.7 Let $V(P_n) = \{u_1, u_2, \dots, u_n\}$ as usual. Then

- (1) $\chi_{td}(P_2) = 2$. The only minimum td-coloring is (given by the color classes) $\{\{u_1\}, \{u_2\}\}$
- (2) $\chi_{td}(P_3) = 2$. The only minimum td-coloring is $\{\{u_1, u_3\}, \{u_2\}\}$.
- (3) $\chi_{td}(P_4) = 3$ with unique minimum coloring $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}\}$.
- (4) $\chi_{td}(P_5) = 4$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4\}, \{u_5\}\}$ or $\{\{u_1, u_5\}, \{u_2\}, \{u_3\}, \{u_4\}\}$ or $\{\{u_1\}, \{u_2\}, \{u_4\}, \{u_3, u_5\}\}$.
- (5) $\chi_{td}(P_6) = 4$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_4, u_6\}, \{u_2\}, \{u_5\}\}$.
- (6) $\chi_{td}(P_7) = 5$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$ or $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$ or $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$.

Proof We prove only (vi). The rest are easy to prove. Now, $\gamma_t(P_7) = \lceil \frac{7}{2} \rceil = 4$. Clearly $\chi_{td}(P_7) \geq 4$. We first show that $\chi_{td}(P_7) \neq 4$. Let C be a td-coloring of P_7 with 4 colors. The vertices u_2 and $u_{-2} = u_6$ must have non-repeated colors. Suppose now that u_3 has a repeated color. Then $\{u_1, u_2, u_3\}$ must contain two color classes and $C|_{\langle [4, 7] \rangle}$ must be a td-coloring which will require at least 3 new colors (by (3)). Hence u_3 and similarly u_{-3} must be non-repeated colors. But, then we require more than 4 colors. Thus $\chi_{td}(P_7) = 5$. Let C be a minimal td-coloring of P_7 . Let u_2 and u_{-2} have colors 1 and 2 respectively. Suppose that both u_3 and u_{-3} are non-repeated colors. Then, we have the coloring $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$. If either u_3 or u_{-3} is a repeated color, then the coloring C can be verified to be equivalent to the coloring given by $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$, or by $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$. \square

We next show that $n = 8, 13, 15, 22$ are good integers.

Theorem 2.8 $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ if $n = 8, 13, 15, 22$.

Proof As usual, we always adopt the convention $V(P_n) = \{u_1, u_2, \dots, u_n\}; u_{-i} = u_{n+1-i}$ for $i \geq \lceil \frac{n}{2} \rceil$; C denotes a minimum td-coloring of P_n .

We have only to prove $|C| > \gamma_t(P_n) + 1$. We consider the following four cases.

Case 1 $n = 8$

Let $|C| = 5$. Then, as before u_2 , being the only vertex dominated by u_1 has a non-repeated color. The same argument is true for u_{-2} also. If now u_3 has a repeated color, $\{u_1, u_2, u_3\}$ contains 2-color classes. As $C|_{\langle [4, 8] \rangle}$ is a td-coloring, we require at least 4 more colors. Hence, u_3 and similarly u_{-3} must have non-repeated colors. Thus, there are 4 singleton color classes and $\{u_2\}, \{u_3\}, \{u_{-2}\}$ and $\{u_{-3}\}$. The two adjacent vertices u_4 and u_{-4} contribute two more colors. Thus $|C|$ has to be 6.

Case 2 $n = 13$

Let $|C| = 8 = \gamma_t(P_{13}) + 1$. As before u_2 and u_{-2} are non-repeated colors. Since $\chi_{td}(P_{10}) = 7 + 2 = 9$, u_3 can not be a repeated color, arguing as in case (i). Thus, u_3 and u_{-3} are also non-repeated colors. Now, if u_1 and u_{-1} have different colors, a diagonal of the color classes chosen

as $\{u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}, \dots\}$ form a totally dominating set of cardinality $8 = \gamma_t(P_{13}) + 1$. However, clearly u_1 and u_{-1} can be omitted from this set without affecting total dominating set giving $\gamma_t(P_{13}) \leq 6$, a contradiction. Thus, u_1 and $u_{-1} = u_{13}$ have the same color say 1. Thus, $\langle [4, -4] \rangle = \langle [4, 10] \rangle$ is colored with 4 colors including the repeated color 1. Now, each of the pair of vertices $\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}$ contains a color classes. Thus $u_9 = u_{-5}$ must be colored with 1. Similarly, u_5 . Now, if $\{u_4, u_6\}$ is not a color class, the vertex with repeated color must be colored with 1 which is not possible, since an adjacent vertex u_5 which also has color 1. Therefore $\{u_4, u_6\}$ is a color class. Similarly $\{u_8, u_{10}\}$ is also a color class. But then, u_7 will not dominate any color class. Thus $|C| = 9$.

Case 3 $n = 15$

Let $|C| = 9$. Arguing as before, u_2, u_{-2}, u_3 and u_{-3} have non-repeated colors $[\chi_{td}(P_{12}) = 8]$; u_1 and u_{-1} have the same color, say 1. The section $\langle [4, -4] \rangle = \langle [4, 12] \rangle$ consisting of 9 vertices is colored by 5 colors including the color 1. An argument similar to the one used in Case (2), gives u_4 (and u_{-4}) must have color 1. Thus, $C| \langle [5, -5] \rangle$ is a td-coloring with 4 colors including 1. Now, the possible minimum td-coloring of P_7 are given by Theorem 2.7. We can check that 1 can not occur in any color class in any of the minimum colorings given. e.g. take the coloring given by $\{u_5, u_8\}, \{u_6\}, \{u_7\}, \{u_9, u_{11}\}, \{u_{10}\}$. If u_6 has color 1, u_5 can not dominate a color class. Since u_4 has color 1, $\{u_5, u_8\}$ can not be color class 1 and so on. Thus $\chi_{td}(P_{15}) = 10$.

Case 4 $n = 22$

Let $|C| = \gamma_t(P_{22}) + 1 = 13$. We note that $\chi_{td}(P_{19}) = \gamma_t(P_{19}) + 2 = 12$. Then, arguing as in previous cases, we get the following facts.

Fact 1 u_2, u_{-2}, u_3, u_{-3} have non-repeated colors.

Fact 2 u_1 and u_{-1} have the same color, say 1.

Fact 3 u_7 is a non-repeated color.

This follows from the facts, otherwise $C| \langle [8, 22] \rangle$ will be a td-coloring; The section $\langle [1, 7] \rangle$ contain 4 color classes which together imply $\chi_{td}(P_{22}) \geq 4 + \chi_{td}(P_{15}) = 4 + 10 = 14$. In particular $\{u_5, u_7\}$ is not a color class.

Fact 4 The Facts 1 and 2, it follows that $C| \langle [4, -4] \rangle = C| \langle [4, 19] \rangle$ is colored with 9 colors including 1. Since each of the pair $\{\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}, \{u_9, u_{11}\}, \{u_{12}, u_{14}\}, \{u_{13}, u_{15}\}, \{u_{16}, u_{18}\}, \{u_{17}, u_{19}\}\}$ contain a color class, if any of these pairs is not a color class, one of the vertices must have a non-repeated color and the other colored with 1. From Fact 3, it then follows that the vertex u_5 must be colored with 1. It follows that $\{u_4, u_6\}$ must be a color class, since otherwise either u_4 or u_6 must be colored with 1.

Since $\{u_4, u_6\}$ is a color class, u_7 must dominate the color class $\{u_8\}$.

We summarize:

- u_2, u_3, u_7, u_8 have non-repeated colors.
- $\{u_4, u_6\}$ is a color class

- u_1 and u_5 are colored with color 1.

Similarly,

- $u_{-2}, u_{-3}, u_{-7}, u_{-8}$ have non-repeated colors.
- $\{u_{-4}, u_{-6}\}$ is a color class.
- u_{-1} and u_{-5} are colored with color 1.

Thus the section $\langle [9, -9] \rangle = \langle [9, 14] \rangle$ must be colored with 3 colors including 1. This is easily seen to be not possible, since for instance this will imply both u_{13} and u_{14} must be colored with color 1. Thus, we arrive at a contradiction. Thus $\chi_{td}(P_{22}) = 14$. \square

Theorem 2.9 *Let n be an integer. Then,*

- (1) *any integer of the form $4k$, $k \geq 2$ is good;*
- (2) *any integer of the form $4k + 1$, $k \geq 3$ is good;*
- (3) *any integer of the form $4k + 2$, $k \geq 5$ is good;*
- (4) *any integer of the form $4k + 3$, $k \geq 3$ is good.*

Proof The integers $n = 2, 3, 6$ are very bad and $n = 4, 5, 7, 9, 10, 11, 14, 18$ are bad. \square

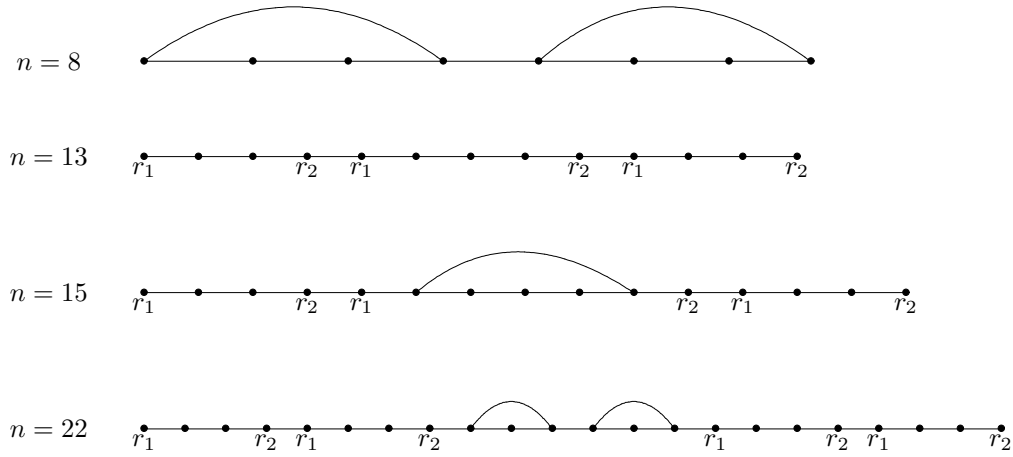
Remark 2.10 Let C be a minimal td-coloring of G . We call a color class in C , a non-dominated color class (n-d color class) if it is not dominated by any vertex of G . These color classes are useful because we can add vertices to these color classes without affecting td-coloring.

Lemma 2.11 *Suppose n is a good number and P_n has a minimal td-coloring in which there are two non-dominated color class. Then the same is true for $n + 4$ also.*

Proof Let C_1, C_2, \dots, C_r be the color classes for P_n where C_1 and C_2 are non-dominated color classes. Suppose u_n does not have color C_1 . Then $C_1 \cup \{u_{n+1}\}, C_2 \cup \{u_{n+4}\}, \{u_{n+2}\}, \{u_{n+3}\}, C_3, C_4, \dots, C_r$ are required color classes for P_{n+4} . i.e. we add a section of 4 vertices with middle vertices having non-repeated colors and end vertices having C_1 and C_2 with the coloring being proper. Further, suppose the minimum coloring for P_n , the end vertices have different colors. Then the same is true for the coloring of P_{n+4} also. If the vertex u_1 of P_n does not have the color C_2 , the new coloring for P_{n+4} has this property. If u_1 has color C_2 , then u_n does not have the color C_2 . Therefore, we can take the first two color classes of P_{n+4} as $C_1 \cup \{u_{n+4}\}$ and $C_2 \cup \{u_{n+1}\}$. \square

Corollary 2.12 *Let n be a good number. Then P_n has a minimal td-coloring in which the end vertices have different colors. [It can be verified that the conclusion of the corollary is true for all $n \neq 3, 4, 11$ and 18].*

Proof We claim that P_n has a minimum td-coloring in which: (1) there are two non-dominated color classes; (2) the end vertices have different colors.

**Fig.1**

Now, it follows from the Lemma 2.11 that (1) and (2) are true for every good integer. \square

Corollary 2.13 *Let n be a good integer. Then, there exists a minimum td-coloring for P_n with two n -d color classes.*

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Degree Splitting Graph on Graceful, Felicitous and Elegant Labeling

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Abstract: We show that the degree splitting graphs of $B_{n,n}; P_n; K_{m,n}; n(k_4 - 3e)I; n(k_4 - 3e)II(b); n(k_4 - e)II$ and $n(k_4 - 2e)II(a)$ are graceful [3]. We prove $C_3 \hat{O} K_{1,n}$ is graceful, felicitous and elegant [2], Also we prove $K_{2,n}$ is felicitous and elegant.

Key Words: Degree splitting graph, graceful graph, elegant graph, felicitous graph, star and path.

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§1. Introduction

Graph labeling methods were introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. For a graph G , the splitting graph $S(G)$ is obtained from G by adding for each vertex v of G , a new vertex v' so that v' is adjacent to every vertex in G .

Let G be a graph with q edges. A graceful labeling of G is an injection from the set of its vertices into the set $\{0, 1, 2, \dots, q\}$ such that the values of the edges are all the numbers from 1 to q , the value of an edge being the absolute value of the difference between the numbers attributed to their end vertices.

In 1981 Chang, Hiu and Rogers defined an elegant labeling of a graph G , with p vertices and q edges as an injective function from the vertices of G to the set $\{0, 1, 2, \dots, q\}$ such that when each edge xy is assigned by the label $f(x) + f(y)(\text{mod } (q + 1))$, the resulting edge labels are distinct and non zero. Note that the elegant labeling is in contrast to the definition of a harmonious labeling [1].

Another generalization of harmonious labeling is felicitous labeling. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \dots, q\}$ is called felicitous labeling if the edge label induced by $f(x) + f(y)(\text{mod } q)$ for each edge xy is distinct.

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§2. Degree Splitting Graph $DS(G)$

Definition 2.1 Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \dots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V \setminus \bigcup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \dots, w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.

§3. Main Theorems

Theorem 3.1 The $DS(B_{n,n})$ is graceful for $n \geq 2$.

Proof Let $G = B_{n,n}$ be a graph. Let $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{uv, uw_2, vw_2\} \cup \{uu_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 4n + 3$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 4n + 3\}$ is as follows:

$f(u) = 1; f(v) = 3; f(w_1) = 0; f(w_2) = 2n + 4; f(u_i) = 4n - 2i + 5$ and $f(v_i) = 2i + 2$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of uv is 2; uu_i is $4n - 2i + 4$ for $1 \leq i \leq n$; vv_i is $2i - 1$ for $1 \leq i \leq n$; w_1u_i is $4n - 2i + 5$ for $1 \leq i \leq n$; w_1v_i is $2i + 2$ for $1 \leq i \leq n$; uw_2 is $2n + 3$ and vw_2 is $2n + 1$. Hence the induced edge labels of $DS(G)$ are $4n + 3$ distinct integers. Hence $DS(G)$ is graceful for $n \geq 2$. \square

Theorem 3.2 The $DS(P_n)$ is graceful for $n \geq 4$.

Proof Let $G = P_n$ be a graph. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_1v_1, w_1v_n\} \cup \{w_2v_i : 2 \leq i \leq n - 1\}$ and $|E(DS(G))| = 2n - 1$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 2n - 1\}$ is as follows:

Case 1 n is odd.

Then $f(w_1) = n + 1; f(w_2) = 0; f(v_i) = i$ for $1 \leq i \leq n$, i is odd and $f(v_i) = 2n - i + 1$ for $1 \leq i \leq n$, i is even.

The corresponding edge labels are as follows:

The edge label of w_2v_i is i for $3 \leq i \leq n - 1$ and i is odd; w_2v_i is $2n - i + 1$ for $1 \leq i \leq n$ and i is even; w_1v_1 is n ; w_1v_n is 1 and $v_i v_{i+1}$ is $2n - 2i$ for $1 \leq i \leq n - 1$. Hence the induced edge labels of $DS(G)$ are $2n - 1$ distinct integers. Hence $DS(G)$ for $n \geq 4$ is graceful.

Case 2 n is even.

The required vertex labeling is as follows:

$f(w_1) = n + 2; f(w_2) = 0; f(v_i) = i$ for $1 \leq i \leq n$, i is odd and $f(v_i) = 2n - i + 1$ for

$1 \leq i \leq n, i$ is even.

The corresponding edge labels are as follows:

The edge label of w_2v_i is i for $3 \leq i \leq n$ and i is odd; w_2v_i is $2n - i + 1$ for $1 \leq i \leq n - 1$ and i is even; w_1v_1 is $n + 1$; w_1v_n is 1 and v_iv_{i+1} is $2n - 2$ for $1 \leq i \leq n - 1$. Hence the induced edge labels of G are $2n - 1$ distinct integers. From case (i) and (ii) the $DS(P_n)$ for $n \geq 4$ is graceful. \square

Theorem 3.3 *The graph $DS(K_{m,n})$ is graceful.*

Proof The proof is divided into two cases following.

Case 1 $m > n$.

Let $G = K_{m,n}$ be a graph. Let $V(G) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_1u_i : 1 \leq i \leq m\} \cup \{w_2v_j : 1 \leq j \leq n\} \cup \{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $|E(DS(G))| = mn + m + n$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, mn + m + n\}$ is as follows:

$f(u_i) = m(n + 2 - i) + n$ for $1 \leq i \leq m$; $f(v_j) = j$ for $1 \leq j \leq n$; $f(w_1) = 0$ and $f(w_2) = n + 1$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $m(n + 2 - i) + n$ for $1 \leq i \leq m$; u_iv_j is $m(n + 2 - i) + n - j$ for $1 \leq i \leq m, 1 \leq j \leq n$ and w_2v_j is $n + 1 - j$ for $1 \leq j \leq n$. Hence the induced edge labels of G are $mn + m + n$ distinct integers. Hence the graph $DS(K_{m,n})$ is graceful.

Case 2 $m = n$.

Let $G = K_{m,n}$ be a graph. Let $V(G) = \{u_i, v_i : 1 \leq i \leq m\}$ and $V(DS(G)) \setminus V(G) = \{w_1\}$. Let $E(DS(G)) = \{w_1u_i, w_1v_i : 1 \leq i \leq m\} \cup \{u_iv_j : 1 \leq i, j \leq m\}$ and $|E(DS(G))| = m(m + 2)$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, m(m + 2)\}$ is as follows:

$f(w_1) = 0$; $f(u_i) = m(m + 3) - mi$ for $1 \leq i \leq m$ and $f(v_i) = i$ for $1 \leq i \leq m$.

The corresponding edge labels are as follows:

The edge label of w_1v_i is i for $1 \leq i \leq m$; u_iv_j is $m(m + 3) - mi - j$ for $1 \leq i, j \leq m$ and w_1u_i is $m(m + 3) - mi$ for $1 \leq i \leq m$. Hence the induced edge labels of G are $m(m + 2)$ distinct integers. Hence the graph $DS(K_{m,n})$ is graceful. \square

Corollary 3.4 *The $DS(K_n)$ is K_{n+1} .*

Theorem 3.5 *The $DS(n(K_4 - 3e)I)$ is graceful.*

Proof Let $G = n(K_4 - 3e)I$ be a graph. Let $V(G) = \{x, y\} \cup \{z_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy\} \cup \{w_1z_i, xz_i, yz_i : 1 \leq i \leq n\}$

and $|E(DS(G))| = 3n + 3$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 3n + 3\}$ is as follows:

$$f(x) = 1; f(y) = 2; f(w_1) = 0; f(w_2) = 4 \text{ and } f(z_i) = 3n + 6 - 3i \text{ for } 1 \leq i \leq n.$$

The corresponding edge labels are as follows:

The edge label of xy is 1; xw_2 is 3; yw_2 is 2; xz_i is $3n + 5 - 3i$ for $1 \leq i \leq n$; yz_i is $3n + 4 - 3i$ for $1 \leq i \leq n$ and w_1z_i is $3n + 6 - 3i$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $3n + 3$ distinct integers. Hence the $DS(G)$ is graceful. \square

Theorem 3.6 *The $DS((n(K_4 - 3e)II(b))$ is graceful.*

Proof Let $G = n(K_4 - 3e)II(b)$ be a graph. Let $V(G) = \{x, y\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_1, xy\} \cup \{w_1v_i, u_iv_i, yu_i, w_2u_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 4n + 2$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 4n + 2\}$ is as follows:

$$f(x) = 3n + 2; f(y) = 1; f(w_1) = 0; f(w_2) = 2; f(v_i) = 3n + 2 + i \text{ for } 1 \leq i \leq n \text{ and } f(u_i) = 2i + 1 \text{ for } 1 \leq i \leq n.$$

The corresponding edge labels are as follows:

The edge label of w_1v_i is $3n + 2 + i$ for $1 \leq i \leq n$; u_iv_i is $3n - i + 1$ for $1 \leq i \leq n$; yu_i is $2i$ for $1 \leq i \leq n$; w_2u_i is $2i - 1$ for $1 \leq i \leq n$; xw_1 is $3n + 2$ and xy is $3n + 1$. Hence the induced edge labels of G are $4n + 2$ distinct integers. Hence the graph $DS(G)$ is graceful. \square

Theorem 3.7 *The $DS(n(K_4 - e)II)$ is graceful.*

Proof Let $G = n(K_4 - e)II$ be a graph. Let $V(G) = \{x, y\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2\} \cup \{w_1u_i, w_1v_i \text{ for } 1 \leq i \leq n\}$ and $|E(DS(G))| = 6n + 3$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 6n + 3\}$ is as follows:

$$f(x) = 0; f(y) = 4n + 2; f(w_1) = 2n + 2; f(w_2) = 4n + 3; f(v_i) = 5n + 4 - i \text{ and } f(u_i) = 6n + 4 - i \text{ for } 1 \leq i \leq n.$$

The corresponding edge labels are as follows:

The edge label of w_1u_i is $4n + 2 - i$ for $1 \leq i \leq n$; w_1v_i is $4n - 1 - i$ for $1 \leq i \leq n$; xw_2 is $4n + 3$; xy is $4n + 2$ and yw_2 is $2n$. Hence the induced edge labels of G are $6n + 3$ distinct integers. The $DS(n(K_4 - e)II)$ is graceful. \square

Theorem 3.8 *The $DS(n(K_4 - 2e)II(a))$ is graceful.*

Proof Let $G = n(K_4 - 2e)II(a)$ be a graph. Let $V(G) = \{x, y, u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xu_i, xy, yu_i, xv_i, v_iw_1, u_iw_2 : 1 \leq i \leq n\}$ and $|E(DS(G))| = 5n + 1$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 5n + 1\}$ is as follows:

$f(x) = 0; f(y) = 2n + 1; f(w_1) = 1; f(w_2) = n + 1; f(v_i) = 5n + 3 - 2i$ and $f(u_i) = 3n + 2 - i$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of xu_i is $3n - i + 2$ for $1 \leq i \leq n$; xy is $2n + 1$; yu_i is $n - i + 1$ for $1 \leq i \leq n$; xv_i is $5n + 3 - 2i$ for $1 \leq i \leq n$; v_iw_1 is $5n + 2 - 2i$ for $1 \leq i \leq n$ and u_iw_2 is $2n - i + 1$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $5n + 1$ distinct integers. The $DS(n(K_4 - 2e)II(a))$ is graceful. \square

Theorem 3.9 *The $DS(C_3\hat{O}K_{1,n})$ is graceful for $n \geq 3$.*

Proof Let graph $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \leq i \leq n\}$ and C_3 be the cycle $xyzx$. Let $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup \{w_1u_i, zu_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 2n + 5$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 2n + 5\}$ is as follows:

$f(w_1) = 1; f(w_2) = 2; f(x) = 4; f(y) = 5; f(z) = 0$ and $f(u_i) = 2i + 5$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of xy is 1; xw_2 is 2; yw_2 is 3; xz is 4; yz is 5; zu_i is $2i + 5$ for $1 \leq i \leq n$ and w_1u_i is $2i + 4$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $2n + 5$ distinct integers. Hence the $DS(G)$ is graceful for $n \geq 3$. \square

Theorem 3.10 *The $DS(C_3\hat{O}K_{1,n})$ is felicitous when $n \geq 3$.*

Proof Let $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \leq i \leq n\}$ and C_3 be the cycle $xyzx$. Let $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup \{w_1u_i, zu_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 2n + 5$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 2n + 5\}$ is as follows:

$f(w_1) = n; f(w_2) = 2n + 2; f(x) = 2n + 3; f(y) = 2n + 4; f(z) = 2n + 5$ and $f(u_i) = i - 1$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The labels of the edges xy is $(4n + 7)(\text{mod } 2n + 5)$; xw_2 is $(4n + 5)(\text{mod } 2n + 5)$; yw_2 is $(4n + 6)(\text{mod } 2n + 5)$; xz is $(4n + 8)(\text{mod } 2n + 5)$; yz is $(4n + 9)(\text{mod } 2n + 5)$; zu_i is $i - 1$ for $1 \leq i \leq n$ and w_1u_i is $n + i - 1$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $2n + 5$ distinct integers. Hence the $DS(C_3\hat{O}K_{1,n})$ is a felicitous for $n \geq 3$. \square

Theorem 3.11 *The $DS(C_3\hat{O}K_{1,n})$ is elegant for $n \geq 3$.*

Proof Let $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \leq i \leq n\}$ and C_3 be the cycle $xyzx$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup$

$\{w_1u_i, zu_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 2n + 5$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 2n + 5\}$ is as follows:

$f(w_1) = n + 2; f(w_2) = 2n + 4; f(x) = n + 3; f(y) = 0; f(z) = 2n + 5$ and $f(u_i) = i + 1$; for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge labels of xy is $(n + 3)$; xw_2 is $(3n + 7)(\text{mod } 2n + 6)$; yw_2 is $(2n + 4)(\text{mod } 2n + 6)$; xz is $(3n + 8)(\text{mod } 2n + 6)$; yz is $(2n + 5)(\text{mod } 2n + 6)$; zu_i is $(2n + 6 + i)(\text{mod } 2n + 6)$ for $1 \leq i \leq n$; w_1u_i is $(n + 3 + i)(\text{mod } 2n + 6)$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $2n + 5$ distinct integers. Hence the $DS(G)$ is elegant for $n \geq 3$. \square

Theorem 3.12 *The $DS(K_{2,n})$ is felicitous for $n \geq 3$.*

Proof Let $G = (K_{2,n})$ be a graph. Let $V(G) = \{v_1, v_2\} \cup \{u_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_2v_1, w_2v_2\} \cup \{w_1u_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 3n + 2$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 3n + 2\}$ is as follows:

$f(u_i) = 3n + 5 - 3i$ for $1 \leq i \leq n$; $f(v_1) = 0$; $f(v_2) = 3n + 1$; $f(w_1) = 1$ and $f(w_2) = 3$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $(3n + 6 - 3i)(\text{mod } 3n + 2)$ for $1 \leq i \leq n$; u_iv_1 is $(3n + 5 - 3i)(\text{mod } 3n + 2)$ for $1 \leq i \leq n$; u_iv_2 is $(6n + 6 - 3i)(\text{mod } 3n + 2)$ for $1 \leq i \leq n$; w_2v_1 is 3 and w_2v_2 is $3n + 4(\text{mod } 3n + 2)$. Hence the induced edge labels of G are $3n + 2$ distinct integers. Hence $DS(K_{2,n})$ is felicitous for $n \geq 3$. \square

Theorem 3.13 *The $DS(K_{2,n})$ is elegant for $n \geq 3$.*

Proof Let $G = (K_{2,n})$ be a graph, Let $V(G) = \{v_1, v_2\} \cup \{u_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_2v_1, w_2v_2\} \cup \{w_1u_i : 1 \leq i \leq n\}$ and $|E(DS(G))| = 3n + 2$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \dots, 3n + 2\}$ is as follows:

$f(u_i) = 3n + 5 - 3i$ for $1 \leq i \leq n$; $f(v_1) = 0$; $f(v_2) = 3n + 1$; $f(w_1) = 2$ and $f(w_2) = 4$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $(3n + 7 - 3i)(\text{mod } 3n + 3)$ for $1 \leq i \leq n$; u_iv_1 is $(3n + 5 - 3i)(\text{mod } 3n + 3)$ for $1 \leq i \leq n$; u_iv_2 is $(6n + 6 - 3i)(\text{mod } 3n + 3)$ for $1 \leq i \leq n$; w_2v_1 is 4 and w_2v_2 is $3n + 5(\text{mod } 3n + 3)$. Hence the induced edge labels of G are $3n + 2$ distinct integers. The $DS(G)$ is elegant for $n \geq 3$. \square

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Distance Two Labeling of Generalized Cacti

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Abstract: A distance two labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. The $L(2, 1)$ -labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2, 1)$ -labeling with $\max\{f(v) : v \in V(G)\} = k$. Here we introduce a new graph family called generalized cactus and investigate the λ -number for the same.

Key Words: Channel assignment, interference, distance two labeling, block, cactus.

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§1. Introduction

In a communication network, the main task is to assign a channel (non negative integer) to each TV or radio transmitters located at different places such that communication do not interfere. This problem is known as channel assignment problem which was introduced by Hale [4]. Usually, the interference between two transmitters is closely related with the geographic location of the transmitters. If we consider two level interference namely major and minor then two transmitters are *very close* if the interference is major while *close* if the interference is minor. Robert [7] proposed a variation of the channel assignment problem in which *close* transmitters must receive different channels and *very close* transmitters must receive channels that are at two apart.

In a graph model of this problem, the transmitters are represented by the vertices of a graph; two vertices are *very close* if they are adjacent and *close* if they are at distance two apart in the graph. Motivated through this problem Griggs and Yeh [3] introduced $L(2, 1)$ -labeling which is defined as follows:

Definition 1.1 A distance two labeling (or $L(2, 1)$ -labeling) of a graph $G = (V(G), E(G))$ is a function f from vertex set $V(G)$ to the set of all nonnegative integers such that the following conditions are satisfied:

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- (1) $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$;
- (2) $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$.

The span of f is defined as $\max\{|f(u) - f(v)| : u, v \in V(G)\}$. The λ -number for a graph G , denoted by $\lambda(G)$ which is the minimum span of a distance two labeling for G . The $L(2, 1)$ -labeling is studied in the past two decades by many researchers like Yeh [14]-[15], Georges and Mauro [2], Sakai [8], Chang and Kuo [1], Kuo and Yan [5], Lu et al. [6], Shao and Yeh [9], Wang [12], Vaidya et al. [10] and by Vaidya and Bantva [11].

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For the graph G , Δ denotes the maximum degree of the graph and $N(v)$ denotes the neighborhood of v . Also in the discussion of distance two labeling $[0, k]$ denotes the set of integers $\{0, 1, \dots, k\}$. For all other standard terminology and notations we refer to West [13]. Now we will state some existing results for ready reference.

Proposition 1.2([1]) $\lambda(H) \leq \lambda(G)$, for any subgraph H of a graph G .

Proposition 1.3([14]) The λ -number of a star $K_{1,\Delta}$ is $\Delta + 1$, where Δ is the maximum degree.

Proposition 1.4([1]) If $\lambda(G) = \Delta + 1$ then $f(v) = 0$ or $\Delta + 1$ for any $\lambda(G)$ - $L(2, 1)$ -labeling f and any vertex v of maximum degree Δ . In this case, $N[v]$ contains at most two vertices of degree Δ , for any vertex $v \in V(G)$.

§2. Main Results

The problem of labeling of trees with a condition at distance two remained the focus of many research papers as its λ -number depends upon the maximum degree of a vertex. In [3], Griggs and Yeh proved that the λ -number of any tree T with maximum degree Δ is either $\Delta + 1$ or $\Delta + 2$. They obtained the λ -number by first-fit greedy algorithm. Later, trees are classified according to their λ -numbers. The trees with λ -number $\Delta + 1$ are classified as *class one* otherwise they are of *class two*. Earlier it was conjectured that the classification problem is NP-complete but Chang and Kuo [1] presented a polynomial time classification algorithm. But even today the classification of trees of class two is an open problem. Motivated through this problem, we present here a graph family which is not a tree but its λ -number is either $\Delta + 1$ or $\Delta + 2$ and it is a super graph of tree.

A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. An n -complete cactus is a simple graph whose all the blocks are isomorphic to K_n . We denote it by $C(K_n)$. An n -complete k -regular cactus is an n -complete cactus in which each cut vertex is exactly in k blocks. We denote it by $C(K_n(k))$. The block which contains only one cut vertex is called leaf block and that cut vertex is known as leaf block cut vertex. We illustrate the definition by means of following example.

Example 2.1 A 3-complete cactus $C(K_3)$ and 3-complete 3-regular cactus are shown in Fig.1 and Fig.2 respectively.

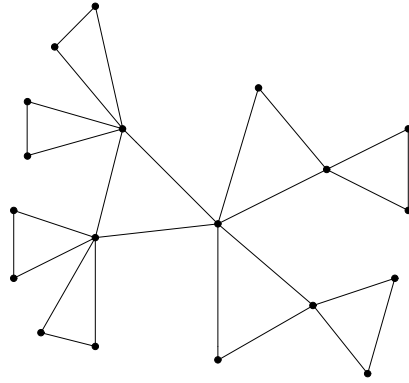


Fig.1

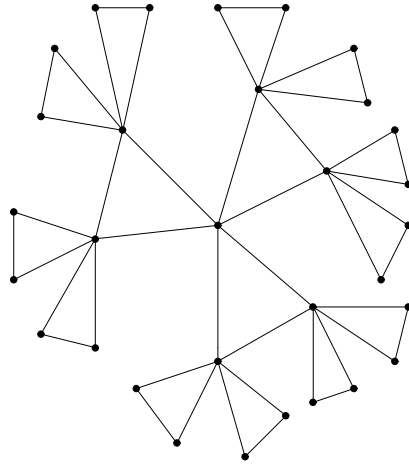


Fig.2

Theorem 2.2 Let $C(K_n(k))$ be an n -complete k -regular cactus with maximum degree Δ and $k \geq 3$. Then $\lambda(C(K_n(k)))$ is either $\Delta + 1$ or $\Delta + 2$.

Proof Let $C(K_n(k))$ be an n -complete k -regular cactus with maximum degree Δ . The star $K_{1,\Delta}$ is a subgraph of $C(K_n(k))$ and hence $\lambda(C(K_n(k))) \geq \Delta + 1$.

For upper bound, we apply the following Algorithm:

Algorithm 2.3 The $L(2,1)$ -labeling of given n -complete k -regular cactus.

Input An n -complete k -regular cactus graph with maximum degree Δ .

Idea Identify the vertices which are at distance one and two apart.

Initialization Let v_0 be the vertex of degree Δ . Label the vertex v_0 by 0 and take $S = \{v_0\}$.

Example 2.4 In Fig.3, the $L(2,1)$ -labeling of 3-complete 3-regular cactus $C(K_3(3))$ is shown for which $\lambda(C(K_3(3))) = \Delta + 2 = 8$.

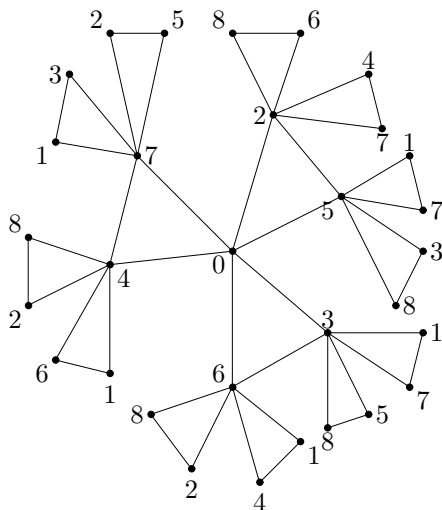
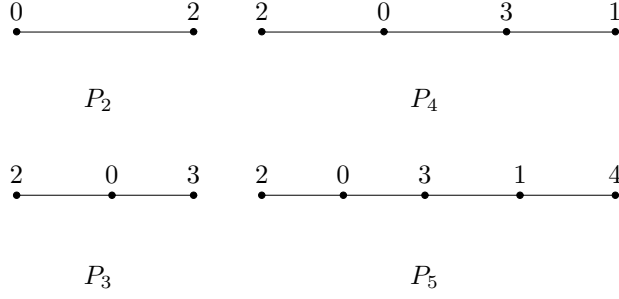


Fig.3

(1) $\lambda(P_2) = 2$;
(2) $\lambda(P_3) = \lambda(P_4) = 3$, and (3) $\lambda(P_n) = 4$, for $n \geq 5$. This can be verified by Proposition 2.2 and using our Algorithm 2.3. In fact, any path P_n is 2-complete 2-regular cactus $C(K_2(2))$. Using a single Algorithm will work to determine the λ -number of path P_n . Using Algorithm 2.3, the $L(2, 1)$ -labeling of P_2, P_3, P_4 and P_5 is demonstrated in Fig.4.

**Fig.4**

Theorem 2.5 *Let $C(K_n)$ be an n -complete cactus with at least one cut vertex which belongs to at least three blocks. Then $\lambda(C(K_n))$ is either $\Delta + 1$ or $\Delta + 2$.*

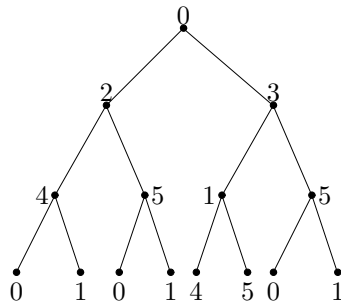
Proof Let $C(K_n)$ be the arbitrary an n -complete cactus with maximum degree Δ . The graph $K_{1,\Delta}$ is a subgraph of $C(K_n)$ and hence by Propositions 1.2 and 1.3 $\lambda(C(K_n)) \geq \Delta + 1$. Moreover $C(K_n)$ is a subgraph of $C(K_n(k))$ (where k is $\frac{\Delta}{n-1}$) and hence by Proposition 1.2 and Theorem 2.2, $\lambda(C(K_n)) \leq \Delta + 2$. Thus, we proved that $\lambda(C(K_n))$ is either $\Delta + 1$ or $\Delta + 2$. \square

Now as a corollary of above result it is easy to show that the λ -number of any tree with maximum degree Δ is either $\Delta + 1$ or $\Delta + 2$. We also present some other graph families as a particular case of above graph families whose λ -number is either $\Delta + 1$ or $\Delta + 2$.

Corollary 2.6 *Let T be a tree with maximum degree $\Delta \geq 2$. Then $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$.*

Proof Let T be a tree with maximum degree $\Delta \geq 2$. If $\Delta = 2$ then T is a path and problem is settled. But if $\Delta > 2$ then $\lambda(T) \geq \Delta + 1$ as $K_{1,\Delta}$ is a subgraph of T . The upper bound of λ -number is $\Delta + 2$ according to Theorem 2.5 as any tree T is a 2-complete cactus $C(K_2)$. Thus, we proved that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$. \square

Example 2.7 In Fig.5, the $L(2, 1)$ -labeling of tree T is shown which is 2-complete cactus $C(K_2)$ with maximum degree $\Delta = 3$ for which $\lambda(T) = \lambda(C(K_2)) = \Delta + 2 = 5$.

**Fig.5**

Corollary 2.8 $\lambda(K_{1,n}) = n + 1$.

Proof The star $K_{1,n}$ is a 2-complete n -regular cactus. Then by Theorem 2.2, $\lambda(K_{1,n}) = n + 1$. \square

Corollary 2.9 For the Friendship graph F_n , $\lambda(F_n) = 2n + 1$.

Proof The Friendship graph F_n is a 3-complete $2n$ -regular cactus. Then by Theorem 2.2, $\lambda(F_n) = 2n + 1$. \square

Example 2.10 In Fig.6 and Fig.7, the $L(2, 1)$ -labeling of star $K_{1,4}$ and Friendship graph F_4 are shown for which λ -number is 5 and 9 respectively.

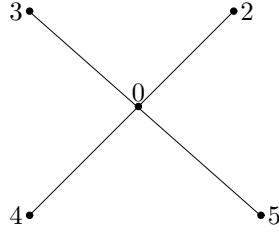


Fig.6

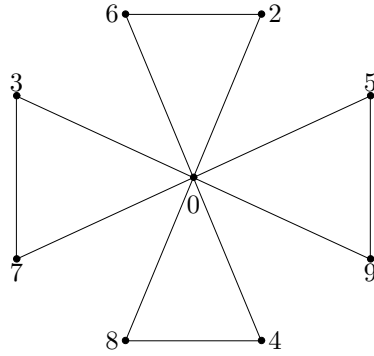


Fig.7

§3. Concluding Remarks

We have achieved the λ -number of an n -complete k -regular cactus. The λ -numbers of some standard graphs determined earlier by Griggs and Yeh [3] can be obtained as particular cases of our results which is the salient feature of our investigations.

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As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

By Albert Einstein, an American theoretical physicist..

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

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[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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